IOP Conf. Series: Materials Science and Engineering **377** (2018) 012042 doi:10.1088/1757-899X/377/1/012042

Free Vibration analysis of bimodular composite material laminated curved beam

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Abstract. In this paper, the free vibration analysis of simply supported bimodular composite material laminated curved beam has been carried out using equivalent stiffness method. The Non-dimensional free vibration frequencies for positive half of vibration cycle and for negative half of vibration cycle have been presented for different ratio and laminated scheme. The analysis is based on classical beam theory. It is observed that the percentage difference of free vibration frequencies obtained from different equivalent stiffness method is more for angle ply laminated beam and for cross ply laminated beam the percentage difference is very less.

1. Introduction

There are some composite materials which exhibit different behavior in tension and compression as shown in figure 1 [1]. These materials are called bimodular composite material. A few examples of such materials are aramid rubber, polyester rubber, carbon-carbon composite, bone and soft tissues etc. The analysis of bimodular material laminated structure is little bit difficult as compare to unimodular composite material laminated structure. Timoshenko [2] was the first person to discuss about the bimodular materials. The major issue of bimodular material is the assignment of material

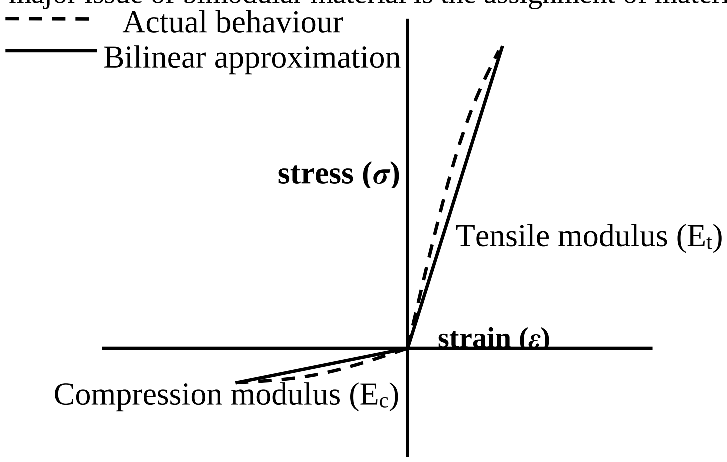

Figure 1. Stress-strain curve of bimodular material

properties which makes the analysis difficult. Jones [1], Bert [3], Papazoglou [4] and Khan *et al.* [5] have suggested different material models to assign proper material properties for the analysis of bimodular material laminated structural element. It has been noticed from the literature available[6-8] in this field that the Bert's material model is used mostly and almost all the analysis were confined to static, stability and free vibration analysis of plates. The free and forced vibration analysis of plates, has been carried out by Patel *et al.* [9]. They have used their own model and have presented a comparative study of Bert's model and their own model. The research paper on bimodular composite material laminated beam is meager. In this paper the free vibration analysis of bimodular composite material laminated curved beam has been carried out using classical beam theory and the stiffness parameters [A, B, D] have been calculated using different equivalent stiffness method.

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2. Methodology

A thin composite simply supported curved beam having radius of curvature R width *b* and thickness *h* as shown in figure 2. The assumed displacement field is:

$$u(x,y,z,t) = u_0(x,t) - z\frac{\partial w(x,t)}{\partial x}$$
$$v(x,y,z,t) = 0 \tag{1}$$
$$w(x,y,z,t) = w(x,t)$$

The strain displacement relation is assumed as:

$$\varepsilon = (\varepsilon_0 + z\kappa) \tag{2}$$

where,

$$\varepsilon_0 = \frac{\partial u_0}{\partial x} + \frac{w_0}{R} \text{ and } \kappa = -\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{R}\frac{\partial u_0}{\partial x}$$

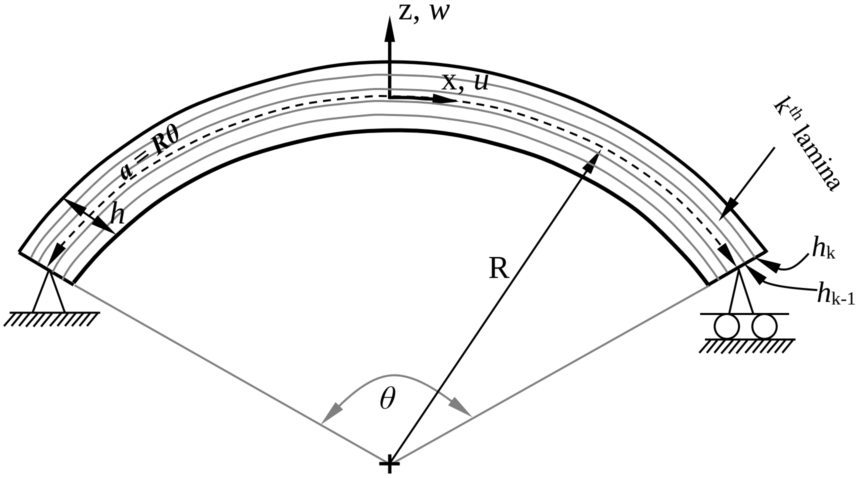

Figure 2. Geometry and coordinate system of simply supported laminated curved beam

Using Hamilton's principles the equation of motions of curved beam can be written as [10]:

$$\frac{\partial N}{\partial x} + \frac{Q}{R} = I_1\frac{\partial^2 u_0}{\partial t^2} - p_x \tag{3}$$

$$-\frac{N}{R} + \frac{\partial Q}{\partial x} = I_1\frac{\partial^2 w_0}{\partial t^2} - p_z \tag{4}$$

where,

$$Q = \frac{\partial M}{\partial x}$$

Where p_x, p_z are external axial and normal forces respectively and for n numbers of lamina and

$$I_1 = \sum_{k=1}^{N_e} b\rho^k (h_k - h_{k-1})$$

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The resultant force and moment are calculated as:

$$[N, M] = b\int_{-h/2}^{h/2}[1, z]\sigma dz \tag{5}$$

The resultant force and moment can be rewritten in terms of various stiffness parameter (A_{11}: Extensional stiffness, B_{11}: Bending stretching coupling stiffness and D_{11}: Bending stiffness) and primary variables as:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}\begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix} \tag{6}$$

2.1. Solution methodology [10]

Considering simply supported boundary condition the general solutions is assumed as:

$$[u_0, w_0] = \sum_{m=1}^{M}[A_m \sin(\alpha_m x), C_m \cos(\alpha_m x)]\sin \omega t \tag{7}$$

where, $\alpha_m = \dfrac{m\pi}{a}$, A_m, C_m are constant, m is integer and a is length of the beam.

The external transverse force(p_z) is expanded in Fourier series and expressed as:

$$p_z = \sum_{m=1}^{M} p_{zm}\cos(\alpha_m x)\sin \omega t, \qquad where,\ p_{zm} = \frac{2}{a}\int p_z \cos(\alpha_m x)dx \tag{8}$$

Using equation (6) to (9), equation (3) and (4) can be rewritten as:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} A_m \\ C_m \end{bmatrix} + \omega^2\begin{bmatrix} I_1 & 0 \\ 0 & -I_1 \end{bmatrix}\begin{bmatrix} A_m \\ C_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

where,

$$C_{11} = -\left(A_{11} + 2B_{11}/R + D_{11}/R^2\right)\alpha_m^2,\ C_{22} = D_{11}\alpha_m^4 + 2(B_{11}/R)\alpha_m^2 + \left(A_{11}/R^2\right)$$

$$C_{21} = -C_{12} = B_{11}\alpha_m^3 + \alpha_m\left[(A_{11}/R) + \left(B_{11}/R^2\right)\right] + (D_{11}/R)\alpha_m^3$$

2.2. The equivalent stiffness method

The stiffness parameters (A_{11}, B_{11}, D_{11}) are calculated using different equivalent stiffness method

2.2.1. Vinson-Sierakowski equivalent stiffness method (VS) [11]

In this method the equivalent stiffness parameters are calculated as:

$$[A_{11}, B_{11}, D_{11}] = \sum_{k=1}^{N_e} bE_x^k\left[(h_k - h_{k-1}), \frac{1}{2}\left(h_k^2 - h_{k-1}^2\right), \frac{1}{3}\left(h_k^3 - h_{k-1}^3\right)\right] \tag{10}$$

where,

$$\frac{1}{E_x^k} = \frac{\cos^4\theta^k}{E_{11n}^k} + \left[\frac{1}{G_{12n}^k} - \frac{2\upsilon_{12n}^k}{E_{11n}^k}\right]\cos^2\theta^k \sin^2\theta^k + \frac{\sin^4\theta^k}{E_{22n}^k}$$

Here, E_{11}^k, E_{22}^k, G_{12}^k, υ_{12}^k and θ^k are young's modulus in fiber direction, young's modulus transverse to fiber direction, in-plane shear deformation, in-plane poison ratio and ply-angle for k^{th} layer respectively.

2.2.2. Rios- Chan equivalent stiffness method (RC) [12]
In this method the entire [ABD] matrix, relates the in-plane force and moment resultants to the strain, is calculated. The relationship among in-plane force and moment resultants and strains can be written as:

$$\left[N_X \;\; N_Y \;\; N_{XY} \;\; M_X \;\; M_Y \;\; M_{XY}\right] = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \left[\varepsilon_X \;\; \varepsilon_Y \;\; \gamma_{XY} \;\; \kappa_X \;\; \kappa_Y \;\; 2\kappa_{XY}\right]^T \tag{11}$$

All components of the [ABD] can be calculated for unimodular material as given [14] and for bimodular materials the required modification is to be done.
The equivalent stiffness parameters are calculated as:

$$A_{11} = \frac{1}{a_{11} - \frac{b_{11}^2}{d_{11}}},\; B_{11} = \frac{1}{b_{11} - \frac{a_{11}d_{11}}{b_{11}}},\; D_{11} = \frac{1}{d_{11} - \frac{b_{11}^2}{a_{11}}} \tag{12}$$

Where, $a_{11}=J_{11}$, $b_{11}=J_{14}$, $d_{11}=J_{44}$ and $[J]=[ABD]^{-1}$
For free vibration the value of axial and normal force are zero and then from above equation we obtained natural frequency. For bimodular material, there are two natural frequency, positive half cycle natural frequency and negative half cycle natural frequency. After obtained the value of natural frequency non-dimensional frequency is obtained by:

$$\left[\Omega_1, \Omega_2\right] = \left[\omega_1, \omega_2\right] a^2 \sqrt{\frac{12\rho}{E_{1c}h^2}} \tag{13}$$

E_{1c} is fibre direction modulus of elasticity in compression.

3. Results and discussion

For the present analysis we have considered aramid rubber and polyester rubber composite materials. The material properties of aramid rubber and polyester rubber are given in table 1. In this analysis a rectangular cross section simply supported curved beam having 1m length *(a)*, 0.025m width *(b)* and 0.01m height *(h)* is considered. The convergence study of non-dimensional positive half cycle frequency for $[0]_4$ laminated bimodular composite has been presented in table 2 for *a/R* =1. It is observed that the converged value of frequency has been obtained in forth iteration. In table 3 the non-dimensional positive and negative half cycle free vibration frequencies for fundamental mode of vibration have been presented for cross-ply laminated beam for different *a/R* ratio. It is observed that the percentage difference of positive and negative half cycle frequencies is very less and also there is no difference between the results of VS and RC methods. The non-dimensional frequency decreases as *a/R* increases for almost all cases for both the materials.

Table 1. Material properties of aramid rubber and polyester rubber [7]

Materials properties	Properties of aramid rubber		Properties of polyester rubber	
	Tension(GPa)	Compression(GPa)	Tension(GPa)	Compression(GPa)
E_1	3.58	0.012	0.617	0.0369
E_2	0.00909	0.012	0.008	0.0106

E_3	0.00909	0.012	0.008	0.0106
G_{12}	0.0037	0.0037	0.00262	0.00267
G_{13}	0.0037	0.0037	0.00262	0.00267
G_{23}	0.0029	0.0029	0.00223	0.00475
ν_{12}	0.416	0.205	0.475	0.185
ν_{13}	0.416	0.205	0.475	0.185
ν_{23}	0.416	0.205	0.475	0.185
ρ	1580	1580	970	970

Table 2. Convergence study of frequency for $[0]_4$ laminated bimodular composite

	Iteration No				
	1	2	3	4	5
Non-dimensional Frequency	54.056	31.027	22.481	20.794	20.794

Table 3. Non-dimensional frequencies for $[0/90]_s$ laminated curved beam for different *a/R* ratio.

	Aramid-Rubber				Polyester-Rubber			
	Ω_1		Ω_2		Ω_1		Ω_2	
a/R	VS	RC	VS	RC	VS	RC	VS	RC
0.0	18.6571	18.6571	18.6572	18.6572	14.1653	14.1789	14.1653	14.1789
0.1	18.6123	18.6124	18.6456	18.6457	14.1344	14.1480	14.1533	14.1669
0.2	18.5136	18.5136	18.5799	18.5799	14.0610	14.0746	14.0985	14.1121
0.3	18.3697	18.3697	18.4686	18.4687	13.9468	13.9602	14.0023	14.0158
0.5	18.0554	18.0555	18.2135	18.2137	13.6070	13.6201	13.6955	13.7087
0.8	18.5320	18.5323	18.6960	18.6966	12.8986	12.9110	13.0242	13.0367
1.0	20.4796	20.4802	20.5780	20.5788	12.3929	12.4047	12.5329	12.5448

Table 4. Non-dimensional frequencies for $[30_2/60_2]$ laminated curved beam for different *a/R* ratio.

	Aramid-Rubber				Polyester-Rubber			
	Ω_1		Ω_2		Ω_1		Ω_2	
a/R	VS	RC	VS	RC	VS	RC	VS	RC
0.0	10.3502	12.5788	8.8227	8.8344	5.1282	6.6606	4.8112	5.3765
0.1	10.3331	12.5601	8.8090	8.8208	5.1199	6.6505	4.8033	5.3678
0.2	10.2849	12.5035	8.7686	8.7804	5.0961	6.6201	4.7810	5.3428
0.3	10.2059	12.4094	8.7021	8.7139	5.0571	6.5697	4.7443	5.3017
0.5	9.9599	12.1139	8.4936	8.5054	4.9354	6.4116	4.6301	5.1735
0.8	9.3942	11.4297	8.0124	8.0238	4.6552	6.0459	4.3672	4.8779
1.0	8.9118	10.8437	7.6006	7.6116	4.4160	5.7329	4.1429	4.6254

In table 4 the non-dimensional frequencies are presented for angle ply laminated beam. The positive and negative half cycle frequencies are not same as cross ply laminated beam. The results obtained from VS and RC methods are not same. The difference between positive and negative half cycle frequencies decreases as a/R increases. The difference between the results obtained from VS and RC method decreases as a/R increases.

4. Conclusion
The following conclusion are drawn

1. For cross-ply laminated beam the positive and negative half cycle frequencies are almost same. The VS and RC method give the same results.
2. For angle-ply laminated beam the positive and negative half cycle frequencies are not same. The VS and RC method give the different results.
3. The difference between positive and negative half cycle frequencies decreases as a/R increases for angle-ply laminated beam.
4. The difference between the results obtained from VS and RC method decreases as *a/R* increases.

5. Symbols and notation

a, b, h, R	Dimensions of curved beam
N, M, Q	Normal force, bending moment and shear force respectively
u, u_0	Axial displacement at arbitrary point and mid-surface respectively
v	Displacement in Y-direction
w, w_0	Transverse displacement at arbitrary point and mid-surface respectively
ε, ε_0	Strain at arbitrary point and mid-surface respectively
κ	Curvature change
ω_1, ω_2	Positive and negative frequency respectively
Ω_1, Ω_2	Positive and negative non-dimensional frequency respectively

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