

# Exact solutions for postbuckling of a graded porous beam

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**Abstract.** An exact, closed-form solution for the postbuckling responses of graded porous beams subjected to axially loading is obtained. It was assumed that the properties of the graded porous materials vary continuously through thickness of the beams, the equations governing the axial and transverse deformations are derived based on the classical beam theory and the physical neutral surface concept. The two equations are reduced to a single nonlinear fourth-order integral-differential equation governing the transverse deformations. The nonlinear equation is directly solved without any use of approximation and a closed-form solution for postbuckled deformation is obtained as a function of the applied load. The exact solutions explicitly describe the nonlinear equilibrium paths of the buckled beam and thus are able to provide insight into deformation problems. Based on the exact solutions obtained herein, the effects of various factors such as porosity distribution pattern, porosity coefficient and boundary conditions on postbuckling behavior of graded porous beams have been investigated.

## 1. Introduction

The present study is devoted to find the exact solution for postbuckling behavior of graded porous beams under the axial compressive load. Since the overwhelming majority of the nonlinear governing equations for large deflection problems are not soluble analytically, it is not easy to find an exact solution to the nonlinear equations for large deflection beams. Only a few exact solutions have been presented up to the present time. However, in the wake of developments in science and technology, the accurate prediction of the postbuckling behavior of beams is an area of concern.

Graded porous materials are a class of lightweight materials in which a change in their material properties is continuous, that is, these properties change with changes in position along one or more directions of the structure to attain a needed purpose. Due to their unique advantages, the application of structures made of graded porous materials is extensive covers a wide range of aerospace, civil and mechanical fields. Therefore, emerging applications of graded porous materials have led to many conducted researches regarding the mechanical behavior of these materials in the past two decades. Magnucki and Stasiewicz [1] explicitly obtained the critical buckling load of a porous beam subjected to the axial compressive load. Jasion and Magnucka-Blandzi [2] investigated the global and local buckling of sandwich metal foam circular plates and beams and the critical buckling loads for these structures were presented and compared with those obtained by other methods. Magnucka-Blandzi and Magnucki [3] gave some optimal parameters in the effective design for a sandwich metal foam beam and the mass and critical load of the beam were considered at the same time in the analysis. Chen *et al* [4] studied the elastic buckling and bending problems for functionally graded porous beams based on shear deformation theory, two porosity distributions were considered in the analysis. The elastic buckling of the aluminum foam beams was investigated by Grygorowicz *et al* [5] and the formula was



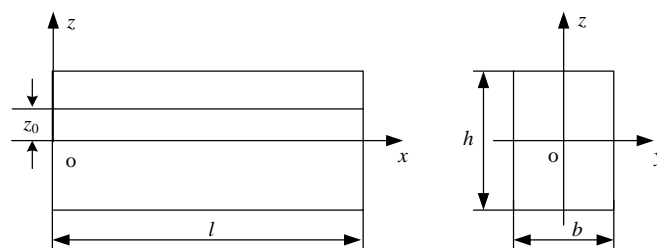
proposed to describe the critical buckling load of the beams. Ebrahimi and Zia [6] used the Galerkin's method to investigate large amplitude vibration of a functionally graded porous beam. Recently, Barati and Zenkour [7] examined the postbuckling behavior of graphene platelet reinforced porous beams and two distributions of graphene platelets were considered in the analysis. Kitipornchai *et al* [8] investigated the elastic buckling and free vibration of graphene platelet reinforced porous beam. The nonlinear vibration and postbuckling responses for graphene platelet reinforced porous beams made of functionally graded materials were investigated by Chen *et al* [9] Shafiei and Kazemi [10] studied the nonlinear buckling behavior of micro- and nano-beams made of functionally graded porous material.

As the aforementioned works show, there are few literatures concerning with an exact solution for postbuckling responses of graded porous beams. Motivated by this consideration, in this study, postbuckling of the classical graded porous beams is investigated. Ma and Lee [11] proposed an exact solution of the postbuckling or bending responses for functionally graded beams and the solution is a function related to the applied load. In the present analysis, their work is extended to the graded porous beams.

In this study, using the physical neutral surface concept, governing equations for the postbuckling response of graded porous beams under axial compressive load are derived based on classical beam theory. The nonlinear equations of equilibrium can be simplified to a single fourth order integral differential equation expressed in term of transverse deformation. The nonlinear equation can be exactly solved, then an exact closed-form solution of the postbuckled configuration is obtained and it is a function related to the axial compressive load. The explicit exact solutions describe the nonlinear buckled paths of the beam and would give ones new insights into these deformation problems. Using the exact solutions, effects of porosity distribution pattern and porosity coefficient on postbuckling responses of graded porous beams have been studied.

## 2. Basic equations

Consider a graded porous material beam with a rectangular cross-section of area  $A$ , height  $h$ , and length  $l$  as shown in figure 1. The origin of the Cartesian coordinate system,  $(x, y, z)$ , is at the left end of the beam in present study. The  $xoy$  plane is placed in the undeformed midplane of the beam, the  $x$  axis is coincident with the centrally axis of the beam, the  $z$  axis is perpendicular to the  $x-y$  plane and positive direction of the  $z$  axis is the one towards the thickness of the cross-section.



**Figure 1.** Geometry and coordinates of a beam.

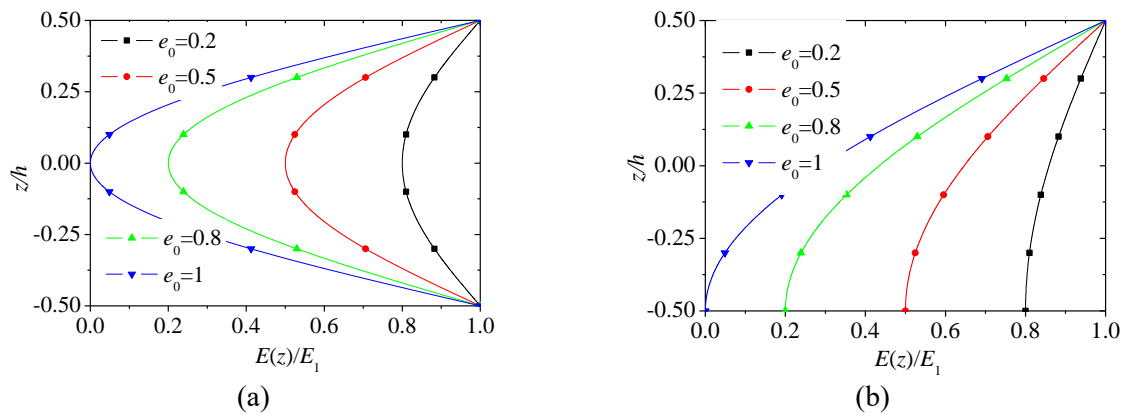
Suppose the mechanical properties of the porous materials vary along the thickness direction of beams. Young's modulus of the porous materials changes according to the two functions, respectively, as follows [12].

$$E(z) = E_1 [1 - e_0 \cos(\pi z/h)] \quad (1)$$

and

$$E(z) = E_1 [1 - e_0 \cos(\pi z/2h + \pi/4)] \quad (2)$$

where,  $e_0$  is the porosity coefficient of the beams,  $e_0 = 1 - E_0/E_1$ ,  $E_1$  denotes Young's modulus of homogeneous materials beams and is also the maximum value of Young's modulus for the graded porous beams which exist on the upper and lower surfaces corresponding to symmetrical porosity distribution, that is, equation (1), and on the top surface corresponding to asymmetrical porosity distribution, i.e., equation (2),  $E_0$  the minimum value on the midplane for symmetrical porosity distribution and on the bottom plane for asymmetrical porosity distribution. The shear modulus  $G(z) = E(z)/[2(1+\nu)]$ , here  $\nu$  is Poisson's ratio and suppose Poisson's ratio is constant along the beam thickness. Figure 2 shows several variation curves of the Young's modulus along thickness of the beam for several values of  $e_0$ , (a): symmetrical porosity distribution, (b): asymmetric porosity distribution.



**Figure 2.** Variations of the Young's modulus along thickness of the beam for several values of  $e_0$ .

When a coordinate system is placed in physical neutral surface of a graded porous beam [13], the term of stretching and bending coupling does not appear in constitutive equations of the beam and this concept can be used to simplify basic equations of the graded porous beam. The physical neutral surface of graded porous beams is represented by  $z = z_0$ .

$$z_0 = \int_{-h/2}^{h/2} z E(z) dz \left( \int_{-h/2}^{h/2} E(z) dz \right)^{-1} \quad (3)$$

Therefore, for symmetrical porosity distribution,  $z_0 = 0$ , and for asymmetric porosity distribution,  $z_0 = h e_0 (4/\pi^2 - 1/\pi) / (1 - 2e_0/\pi)$ . It is obvious that the physical neutral surface is coincident with the geometric midplane for a homogeneous beam or a graded porous beam with symmetrical porosity distribution.

Make use of the physical neutral surface concept and the classical beam theory (CBT), the displacements can be written in the following form

$$U_x(x, z) = u(x) - (z - z_0) \frac{dw}{dx} \quad (4)$$

$$U_z(x, z) = w(x)$$

Here,  $u$  and  $w$  are the displacements of the physical neutral surface along  $x$  and  $z$ , respectively. Axial strain is as follows.

$$\varepsilon_x = \varepsilon_x^0 - (z - z_0) \varepsilon_x^1 = \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] - (z - z_0) \frac{d^2 w}{dx^2} \quad (5)$$

In the above,  $\varepsilon_x^0$  and  $\varepsilon_x^1$  are the strain and curvature in the physical neutral surface, respectively. The constitutive relations are expressed as

$$N_x = \int_A \sigma_x dA = A_x \varepsilon_x^0 + B_x \varepsilon_x^1 = A_x \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] \quad (6)$$

$$M_x = \int_A \sigma_x (z - z_0) dA = B_x \varepsilon_x^0 - D_x \varepsilon_x^1 = -D_x \frac{d^2 w}{dx^2}$$

Here,  $A_x = \int_A E(z) dA$ ,  $B_x = \int_A (z - z_0) E(z) dA = 0$ ,  $D_x = \int_A (z - z_0)^2 E(z) dA$ .

It is obvious that the term of stretching and bending coupling does not appear in equations (6) based on the physical neutral surface theory.

Through the application of energy principle, the equilibrium equations and boundary conditions can be derived as followed based on CBT.

$$\frac{d}{dx} \left\{ A_x \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] \right\} = 0 \quad (7)$$

$$-D_x \frac{d^4 w}{dx^4} + A_x \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] \frac{d^2 w}{dx^2} = 0 \quad (8)$$

$$w = 0 \text{ and } \frac{dw}{dx} = 0 \text{ for a clamped end, } w = 0 \text{ and } \frac{d^2 w}{dx^2} = 0 \text{ for a hinged end} \quad (9)$$

The boundary conditions of the axial displacement  $u$  can be expressed as

$$u(0) = 0 \text{ and } u(l) = -pl/A_x \quad (10)$$

where,  $p$  is an axial compressive load.

### 3. Solution

Integrating equation (7) twice, one obtains:

$$u = -\frac{1}{2} \int_0^x \left( \frac{dw}{d\eta} \right)^2 d\eta + \frac{C_1}{A_x} x + C_2 \quad (11)$$

Using the boundary conditions for the axial displacement, one can obtain:

$$C_1 = \frac{A_x}{2l} \int_0^l \left( \frac{dw}{dx} \right)^2 dx - p, \quad C_2 = 0 \quad (12)$$

The substitution of equations (11) and (12) into equation (8) leads to:

$$-D_x \frac{d^4 w}{dx^4} + \left[ \frac{A_x}{2l} \int_0^l \left( \frac{dw}{dx} \right)^2 dx - p \right] \frac{d^2 w}{dx^2} = 0 \quad (13)$$

Equation (13) is the geometrically nonlinear governing equation of a graded porous beam and in the equation, the midplane stretching is taken account of.

The dimensionless variables are defined as follows

$$\xi = x/l, \quad W = w/h, \quad F_1 = A_x h^2 / D_x, \quad P = p l^2 / D_x, \quad P_0 = p l^2 / D_0$$

where  $D_0 = \int_A z^2 E_1 dA$ . Hence equation (13) can be transformed into the one in dimensionless form:

$$\frac{d^4 W}{d\xi^4} + \lambda^2 \frac{d^2 W}{d\xi^2} = 0 \quad (14)$$

where  $\lambda^2$  is defined by

$$\lambda^2 = P - \frac{F_1}{2} \int_0^1 \left( \frac{dW}{d\xi} \right)^2 d\xi \quad (15)$$

The boundary conditions (9) can be written as

$$W(0) = W(1) = 0 \quad \text{and} \quad \frac{dW(0)}{d\xi} = \frac{dW(1)}{d\xi} = 0 \quad \text{for a beam with clamped ends} \quad (16)$$

$$W(0) = W(1) = 0 \quad \text{and} \quad \frac{d^2 W(0)}{d\xi^2} = \frac{d^2 W(1)}{d\xi^2} = 0 \quad \text{for a simply supported beam}$$

For any given deformed configuration  $W(\xi)$ , the integral of  $W(\xi)$  in equations (14) and (15) is constant, and so a closed form solution for the deformed configuration may be obtained. The closed form solution for the deformed configuration of beams is given by

$$W(\xi) = a \left\{ \frac{1 - \cos \lambda}{\lambda - \sin \lambda} [\sin(\lambda \xi) - \lambda \xi] - \cos(\lambda \xi) + 1 \right\} \quad \text{for a beam with clamped ends} \quad (17)$$

$$W(\xi) = a \sin(\lambda \xi) \quad \text{for a beam with simply supported ends} \quad (18)$$

Here,  $a$  is a constant associated with the axial compressive load  $P$  and can be expressed as:

$$a = \pm \frac{2}{\sqrt{F_1}} \left( \frac{P}{\lambda^2} - 1 \right)^{1/2} \quad (19)$$

The characteristic equation for  $\lambda$  of beams is given by:

$$2 - 2 \cos \lambda - \lambda \sin \lambda = 0 \quad \text{for a beam with clamped ends} \quad (20)$$

$$\sin \lambda = 0 \quad \text{for a beam with simply supported ends} \quad (21)$$

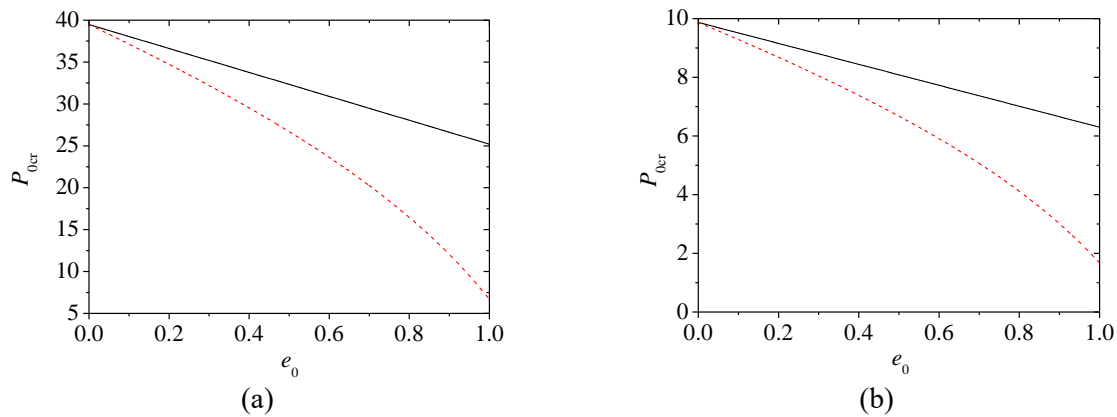
In the early part of buckling, configuration of a buckled beam is extremely approximate to original straight state of the beam. At the present moment, the axially load  $P_0$  is exactly the critical buckling load  $P_{0cr}$  that can be obtained from equation (19) by letting  $a=0$ ,

$$P_{0cr} = \lambda_1^2 C_{e0} \quad (22)$$

where  $C_{e0}$  is a constant related to the porosity coefficient,  $e_0$ , and  $\lambda_1$  denotes the lowest eigenvalue in equation (20) or (21). When  $e_0 = 0$ , the dimensionless critical buckling load of a homogeneous beam is

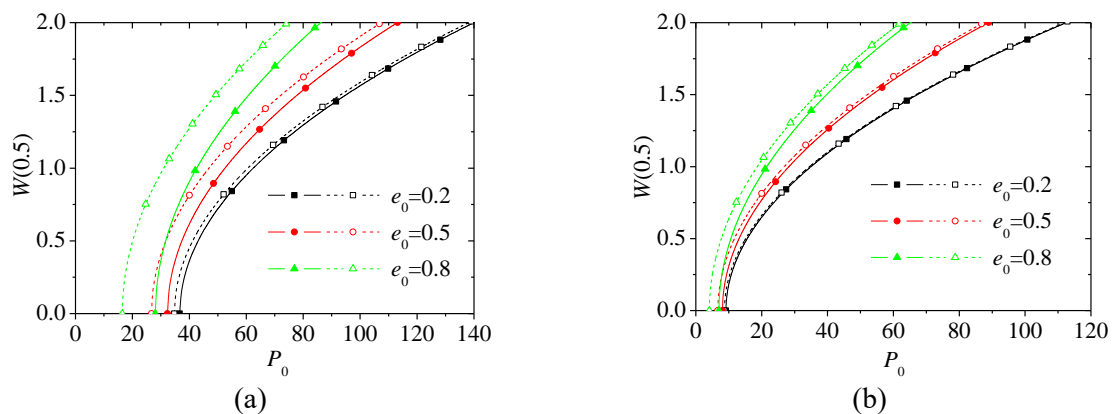
obtained, that is,  $P_{0cr} = 4\pi^2$  for a clamped beam at two ends,  $P_{0cr} = \pi^2$  for a simply supported beam at two ends, which are identical to the solution obtained by Ma and Lee [11].

The variations of the dimensionless critical buckling load  $P_{0cr}$  with the porosity coefficient  $e_0$  are given in figure 3. The solid and dashed lines indicate the results of beams with symmetric and asymmetric porosity distribution, respectively. It is seen from figure 3 that as the porosity coefficient  $e_0$  increases, the critical buckling load decreases. Such a trend is observed because an increase in  $e_0$  results in a decrease in the Young's modulus of graded porous beams.



**Figure 3.** Variation curves of  $P_{0cr}$  vs  $e_0$ . (a): a clamped beam, (b): a simply supported beam.

The typical post-buckling paths of graded porous beams are shown in figure 4. The solid and dashed lines indicate the results of beams with symmetric and asymmetric porosity distribution, respectively. As expected, in figure 4, the variation curves of the post-buckling deflection with axial compressive load for graded porous beams are quite similar to those for pure material beams.



**Figure 4.** Post-buckling paths of a graded porous beam. (a): a clamped beam, (b): a simply supported beam.

#### 4. Conclusion

An exact solution of the postbuckling responses of graded porous beams under the axial compressive load is obtained. The postbuckled configuration of beams was expressed as a function related to axial compressive load. The nonlinear equilibrium paths of the postbuckled beams can be described explicitly by the exact solutions and thus the exact solutions would give ones new insights into

deformation problems and can be used as a standard with which various approximate theories and numerical methods can be verified and improved. One can arrive at some conclusions as followed.

- Under axial compressive load, clamped and simple supported porous beams, in the case of both symmetrical and asymmetric porosity distribution, all exhibit typical bifurcation buckling behavior.
- As the porosity coefficient increases, the critical buckling load decreases, which becomes more pronounced for the case of asymmetric porosity distribution.

### Acknowledgments

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