IOP Conf. Series: Materials Science and Engineering **371** (2018) 012041 doi:10.1088/1757-899X/371/1/012041

Support Draft Calculation for a Ramp in the Form of Developable Helicoid

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Abstract. The article is about the analytical method of calculation a ramp in the form of developable helicoid on the support draft. The asymptotic method of small parameter is applied to solve the system of three differential equilibrium equations for developable helicoid stress-strain. The numerical results of displacements and bending moments are verified and coincide with engineering practice. The suggested approach can be extended for calculation of torso-helicoids with other boundary conditions.

1. Introduction

According to [1] the helical ruled surfaces may be divided into five types: oblique helicoids; right helicoids (a particular case of oblique helicoids, when inclination angle of a generatrix is equal to zero); developable helicoids; convolute helicoids; pseudo-developable helicoids (or right convolute helicoid that is a particular case of convolute helicoid). Developable helicoids can also be called evolvent helicoids, for example in [2], or torso-helicoids, for example in [3], or involute helicoids [4].

Developable helicoids are mostly used in mechanical engineering in metal elements of small sizes [5], [6], where plastic deformations are needed to be taken into account [7]; however, they can also be successfully used in architecture and civil engineering for designing ramps, helical elements, helical parts of car interchanges, geometric models of embankment slopes when the road is raised and rounded (as a surface of the equal slope) [8]; and even as parts of the whole buildings, as it was done in the building of the famous Guggenheim's museum of modern art in New York (USA) by an architect Frank Lloyd Wright [9].

In this paper, there is an attempt to solve a particular practical task by the means of analytical methods of calculation applied to developable helicoids.

A developable helicoid (figure 1) is a developable surface formed by tangents to the helical line of a constant step on a circular cylinder of radius a. The development of the developable helicoid on the plane is an annular region bounded by coaxial circles [10].

Therefore we can use the theory of thin elastic shells and calculate the ramp in the form of a long developable helicoid using the asymptotic method of a small parameter. This method is described in more details in the papers [11], [12]. The aim of this study is to show the analytical approach to the ramps in the form of developable helicoids calculation on a support draft.

2. Asymptotic Method of Small Parameter for Developable Helicoid

The most popular parametric equations for developable helicoids (figure 1) are the following [10]:

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$$x = x(u,v) = a\cos v - \frac{au\sin v}{m},$$
$$y = y(u,v) = a\sin v + \frac{au\cos v}{m},$$
$$z = z(u,v) = bv + \frac{bu}{m}, \quad (1)$$

where $m = \sqrt{a^2 + b^2}$,
a is radius of the cylinder, tangents to which are the generators of a developable helicoids;
b is a hitch of a curve $u = 0$ (return edge);
v is an angle deflected from axis *Ox*.

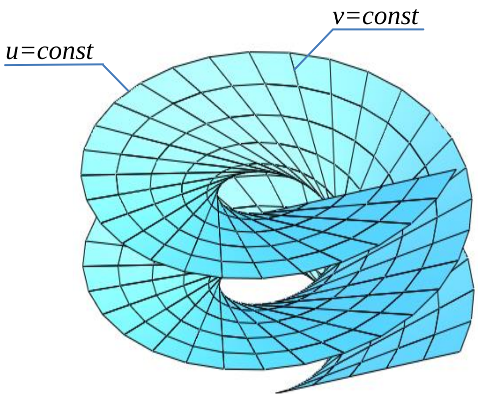

Figure 1. Developable helicoid

We determine the equations of developable helicoids as follows

$$x = x(u,s) = a_0\cos^2\varphi\left(\cos\frac{s}{m} - \frac{u}{m}\sin\frac{s}{m}\right),$$
$$y = y(u,s) = a_0\cos^2\varphi\left(\sin\frac{s}{m} + \frac{u}{m}\cos\frac{s}{m}\right),$$
$$z = z(u,s) = (s+u)\sin\varphi, \quad (2)$$

Here *u*, *s* are non-orthogonal conjugate coordinate lines (the coordinate lines *u* coincide with the rectilinear generatrixes of the torso-helicoid, and the lines *s* are equidistant helical lines of the surface);
$m = a_0\cos\varphi$;
$a = a_0\cos^2\varphi$;
$b = a_0\sin\varphi\cos\varphi$;
a_0 is a radius of a helical evolvent;
φ – is inclination angle between the generatrix and the plane;
$tg\varphi = b/a$;
$s = mv$;
s is an arc of a helical cuspidal edge.
The asymptotic method of small parameter in application to developable helicoids was introduced by Krivoshapko in [11] and developed by Rynkovskaya in [12], [13].
We can use here the system of three ordinary differential equations in dimensionless displacements *U*, *V*, *W* obtained by Krivoshapko [11] for calculation of thin elastic shells in the form of long torso-helicoids subject to the dead load:

$$\frac{d}{d\alpha}\left[\alpha^3\frac{d}{d\alpha}\left(\frac{U}{\alpha}\right)\right]=-\frac{a_0\alpha^4}{CB^2}X+\mu\left[W-\frac{\alpha}{B^2}\left(1+v\alpha^2\right)\frac{dW}{d\alpha}\right]=-\frac{a_0\alpha^4}{CB^2}X+\mu E,$$

$$\frac{B^4}{\alpha}\frac{dV}{d\alpha}=-\frac{B^2}{\alpha}\frac{dU}{d\alpha}+2U+2\mu W-\frac{2a_0}{(1-v)C}\int\alpha\left(BY+X\right)d\alpha+A_1,$$

$$\frac{d}{d\alpha}\left\{\frac{t}{\alpha}\frac{d}{d\alpha}\left[\frac{B^4}{\alpha}\frac{d}{d\alpha}\left(\frac{1}{\alpha}\frac{dW}{d\alpha}\right)\right]+\mu\frac{1-v}{2\alpha}\left(\frac{dU}{d\alpha}+B^2\frac{dV}{d\alpha}\right)+\mu\alpha\frac{dU}{d\alpha}+\mu v\left(U+\mu W\right)\right\}+\frac{a_0\alpha}{C}\left(\mu X+\mu\frac{Y}{B}-Z\right)=0. \quad (3)$$

U, V, W are dimensionless parameters of displacements;
X, Y, Z are components of surface distributed load (analogue to dead load);

$$\alpha=\frac{u}{a_0};\ U=\frac{U_u}{a_0};\ V=\frac{U_s}{a_0B};\ W=\frac{U_z}{a_0};\ \mu=tg\varphi;\ t=\frac{h^2}{12{a_0}^2}; \quad (4)$$

where U_u, U_s, U_z are vector components of elastic displacement of the shell middle surface (with the width h).
Dimensionless parameters of displacements U, V, W may be expressed from Eq. 3 as follows:

$$U=\mu\frac{a_0Z}{C}\left(\frac{\alpha^2}{3}+\frac{1}{2}-\frac{B^2}{2\alpha}arctg\alpha\right)+\mu\alpha K_2+\frac{A_2}{\alpha}+\alpha A_3,$$

$$E=\left[W-\frac{\alpha}{B^2}\left(1+v\alpha^2\right)\frac{dW}{d\alpha}\right],$$

$$V=\mu\int\left[\left(\alpha^2-1\right)\frac{K_2}{B^4}-\frac{K_1}{\alpha^2B^2}+\frac{2\alpha}{B^4}W\right]d\alpha+\mu\frac{a_0}{2C}\left[\frac{1}{B^2(1-v)}-\frac{1}{3B^2}+\frac{\ln B^2}{1-v}+\right.$$

$$\left.+\frac{1}{\alpha}arctg\alpha\right]Z-\frac{A_1}{2B^2}-\frac{A_2}{\alpha B^2}-\frac{\alpha}{B^2}A_3+A_4,$$

$$tW=\frac{a_0Z}{64C}\alpha^4+\mu\left\{\frac{(1-v)}{32}A_1\left(\ln B^2\right)^2+\frac{(1-v)}{4}\left(A_2-A_3\right)\left[arctg\alpha\left(\ln B^2+\frac{7}{2}+\frac{\alpha^2}{2}\right)-\right.\right.$$

$$\left.\left.-\frac{7}{2}\alpha-\int\frac{\ln B^2}{B^2}d\alpha\right]+\frac{(1+v)}{3}A_3\left[\frac{arctg\alpha}{4}\left(3\alpha^2+5\right)-\frac{5}{4}\alpha-\frac{\alpha^3}{3}\right]\right\}+$$

$$+\mu^2\frac{a_0Z}{6C}\left\{(1+v)\left[\alpha arctg\alpha\left(\frac{\alpha^2}{3}+\frac{5}{4}\right)-\frac{1}{8}(arctg\alpha)^2\left(5+3\alpha^2\right)\right]-(2v+1)\frac{\alpha^4}{32}-\right. \quad (5)$$

$$\left.-\frac{5}{16}\left(\ln B^2\right)^2\right\}+\mu^2\int\left\{\alpha\int\left[\frac{\alpha}{B^4}\int\left(\frac{1-v}{B^2}\alpha^4K_2+\alpha^2K_2-\frac{1+v\alpha^2}{B^2}\alpha W\right)d\alpha-\right.\right.$$

$$\left.\left.-\frac{\alpha^4}{B^4}K_2\right]d\alpha\right\}d\alpha+C_1B^2\ln B^2+C_2\ln B^2+C_3\alpha^2+C_4$$

Since we use here the method of small parameter, the inclination angle between the generatrix and the plane must be $\varphi < 45^0$ ($\mu = tg\varphi < 1$), and solutions U, V, W are found in the form of series in powers of the small parameter μ:

$$U=U(\alpha,\mu)=\sum_{k=0}^{\infty}U_k(\alpha)\mu^k,$$

$$V=V(\alpha,\mu)=\sum_{k=0}^{\infty}V_k(\alpha)\mu^k,$$

$$W=W(\alpha,\mu)=\sum_{k=0}^{\infty}W_k(\alpha)\mu^k, \quad (6)$$

where U_k, V_k, W_k are vector coefficients to be found.

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The small parameter method is used to solve mechanical and physical problems with differential equations containing certain parameters in cases when it is possible to find a particular solution of the differential equation for certain fixed values of these parameters that satisfy the initial conditions [14]. Substituting expressions (Eq. 6) into equations (Eq. 5), and equating consecutively free members and coefficients for powers μ, we obtain a system of $(3+3k)$ equations for computing U_k, V_k, W_k. For example, to determine the vector coefficients U_0, V_0, W_0, it is necessary to take in equations (Eq.5) that $\mu = 0$. As a result, we find solutions of the generating equations:

$$U_0 = \frac{A_{20}}{\alpha} + \alpha A_{30},$$

$$V_0 = -\frac{A_{10}}{2B^2} - \frac{A_{20}}{\alpha B^2} - \frac{\alpha}{B^2} A_{30} + A_{40},$$

$$W_0 = a_0{}^3 Z\alpha^4 / (64D) + C_{10}B^2 \ln B^2 + C_{20} \ln B^2 + C_{30}\alpha^2 + C_{40}. \quad (7)$$

3. Application for calculation of a support draft

We consider a long shallow developable helicoid with the following geometric characteristics (in the form of parametric equations Eq.1): $a = 3$ m; $b = 0{,}2$ m; $u_1 = 4$ m; $u_2 = 6$ m; h=0,01 m.

If we use the parametric equations Eq.2, these characteristics correspond to the slope angle of the rectilinear generatrix $\varphi = 0{,}06657\ rad = 3{,}81418\ ^0$; the inner radius $R1 = 4{,}993$ m, the external radius $R2 = 6{,}696$ m, the radius of the a radius of a helical evolvent of the torso-helicoid return to the plane $a_0 = 3.0133$ m.

Let us determine modulus of elasticity $E = 2 \cdot 10^8\ kPa$ and Poisson's ration $\nu = 0{,}3$.

We will assume that the surface load is $P_x = P_y = P_z = 0$, therefore, $X = Y = Z = 0$; and determine that the curvilinear edges $\alpha = \alpha_1$ and $\alpha = \alpha_2$ are rigidly clamped.

Let us calculate the stress-strain state of the developable helicoid when the internal curvilinear contour $\alpha = \alpha_1$ is displaced along the fixed *z*-axis by

$$\delta = \frac{u_z m^2}{a} = \frac{u_z \sqrt{a^2 + b^2}}{a}. \quad (8)$$

We will assume $u_z(\alpha = \alpha_1) = 0.005$ m.

For a developable helicoid determined by parametric equations Eq.1, when the return-edge equations of the developable helicoid are written in the form

$$x = a\cos v,$$
$$y = a\sin v,$$
$$z = bv, \quad (9)$$

angles between the fixed axis *z* and the directions of displacements u_u, u_v, u_z are

$$\cos(z_н \wedge u_u) = \frac{z'}{F} = \frac{b}{F},$$

$$\cos(z_н \wedge z_n) = \cos(z_н \wedge u_z) = \frac{a}{F}, \quad (10)$$

where $F = \sqrt{a^2 + b^2}$.

The angle between the tangent to the coordinate line *v* and the fixed axis *z* is found from the vector expression

$$e_v = \frac{r_v}{B} = \cos(x_н \wedge u_v)i + \cos(y_н \wedge u_v)j + \cos(z_н \wedge u_v)k,$$

where e_v is the unit vector that is tangent to the coordinate line *v*;

$r_v = \frac{\partial r}{\partial v}$, where *r* is a radius vector; and the vector equation of the surface of the developable helicoids is

$$r = r(u, v) = \rho(v) + ul(v),$$

where $\rho(v)$ is a current radius vector of a return edge:

$$\rho(v) = x(v)i + y(v)j + z(v)k,$$

$l(v)$ is the unit tangent vector given at each point of the return edge:

$$l(v) = \frac{x'(v)i + y'(v)j + z'(v)k}{\sqrt{x'^2(v) + y'^2(v) + z'^2(v)}}.$$

Then $\cos(z_н \wedge u_v) = \dfrac{b}{B}$, where $B = \sqrt{F^2 + u^2 a^2 / F^2}$.

However, for the shape of the developable helicoid in the parametric form (Eq. 2), when the length of the arc *s* of the return edge is taken as the parameter *v*, the formulas for the angles between the fixed *z* axis and the displacement directions u_u, u_s, u_z (Eq. 10) can be written in the following form:

$$\cos(z_н \wedge u_u) = \sin\varphi,$$

$$\cos(z_н \wedge z_n) = \cos(z_н \wedge u_z) = \cos\varphi. \tag{11}$$

The angle between the tangent to the arc length of the return edge *s* and the fixed axis *z* is found from the vector expression

$$e_s = \frac{r_s}{B} = \cos(x_н \wedge u_s)i + \cos(y_н \wedge u_s)j + \cos(z_н \wedge u_s)k,$$

where $r = r(u,s) = \rho(s) + ul(s)$,

$\rho(s) = x(s)i + y(s)j + z(s)k$,

$l(s) = \rho'(s) = x'(s)i + y'(s)j + z'(s)k$.

We obtain here

$$\cos(z_н \wedge u_s) = \frac{a_0 \sin\varphi}{\sqrt{a_0^2 + u^2}},$$

$$\cos(z_н \wedge u_s) = \frac{\sin\varphi}{\sqrt{1+\alpha^2}} = \frac{\sin\varphi}{B}, \tag{12}$$

$$B = \sqrt{1+\alpha^2}. \tag{13}$$

The boundary conditions for this task will be written as:

$$W_i(\alpha = \alpha_2) = 0,\ W'_i(\alpha = \alpha_2) = 0,\ U_i(\alpha = \alpha_2) = 0,\ V_i(\alpha = \alpha_2) = 0, \tag{14}$$

$$W_i(\alpha = \alpha_1) = \frac{u_z}{a_0} = \frac{\delta\cos(z_н \wedge u_z)}{a_0} = \frac{\delta\cos\varphi}{a_0},$$

$$W'_i(\alpha = \alpha_1) = 0,$$

$$U_i(\alpha = \alpha_1) = \frac{u_u}{a_0} = \frac{\delta\cos(z_н \wedge u_u)}{a_0} = \frac{\delta\sin\varphi}{a_0},$$

$$V_i(\alpha = \alpha_1) = \frac{u_s}{a_0 B} = \frac{\delta\cos(z_н \wedge u_s)}{a_0 B} = \frac{\delta\sin\varphi}{a_0 B}, \tag{15}$$

where *B* is determined in Eq. 13.

Equations (14) show that the curvilinear outer edge $\alpha = \alpha_2 = const$ of the developable helicoid is fixed and rigidly clamped. Equations (15) allow the rigidly constrained curvilinear contour $\alpha = \alpha_1 = const$ move along the fixed *z* axis by a value *δ*.

To solve the task, we can apply the method of a small parameter, for instance with the first terms of the series

$$U = U_0,\ V = V_0,\ W = W_0.$$

Dimensionless parameters of the first terms of the series U_0, V_0, W_0 are obtained from Eq. 7 as follows:

$$U_0 = \frac{A_{20}}{\alpha} + \alpha A_{30},$$

$$V_0 = -\frac{A_{10}}{2B^2} - \frac{A_{20}}{\alpha B^2} - \frac{\alpha}{B^2}A_{30} + A_{40},$$

$$W_0 = C_{10}B^2 \ln B^2 + C_{20}\ln B^2 + C_{30}\alpha^2 + C_{40}. \tag{16}$$

To calculate the constants $C_{10}, C_{20}, C_{30}, C_{40}$, we apply the boundary conditions Eq. 14, 15 as follows:

$$W_0 = \frac{\partial W_0}{\partial u} = 0 \text{, when } \alpha = \alpha_2 = \frac{u_2}{a_0} \text{, and } u_2 = 6 \text{ m;}$$

$$W_0 = \frac{\delta \cos\varphi}{a_0},\ \frac{\partial W_0}{\partial u} = 0 \text{, when } \alpha = \alpha_1 = \frac{u_1}{a_0} \text{, and } u_1 = 4 \text{ m.}$$

To calculate the constants $A_{10}, A_{20}, A_{30}, A_{40}$, we use the following conditions

$$U_0(\alpha = \alpha_2) = 0,$$

$$V_0(\alpha = \alpha_2) = 0,$$

$$U_0(\alpha = \alpha_1) = \frac{\delta \sin\varphi}{a_0},$$

$$V_0(\alpha = \alpha_1) = \frac{\delta \sin\varphi}{a_0 B}.$$

Finally, for the first term in the series, the expressions for the bending moments in this method will be written in the following form:

$$M_u = -\frac{D}{a_0 \alpha} \frac{d}{d\alpha}\left[\frac{B^2}{\alpha}\frac{dW_0}{d\alpha} - (1-v)W_0\right]. \tag{17}$$

4. Test numerical experiments and discussion

It is necessary to point out that this task is solved in the "pseudo-moments" M_u* (the asterisk ... * in Eq. 17 is not shown conditionally), because the equilibrium equations suggested by Goldenveiser A.G. [15] are used. There is an interrelation between "pseudo-moments" and traditional engineering moments, and it can be converted into traditional internal moments M_u, using the formula:

$$M_u = -M_u{}^* - M_{us}{}^* \cos\chi, \tag{18}$$

where $\cos\chi = \frac{F}{AB} = \frac{1}{B} = \frac{1}{\sqrt{1+\alpha^2}}$, and χ is an angle between coordinate lines u and v.

As it is shown in Fig. 2, the normal displacements are distributed in accordance with engineering practice, and the boundary conditions are satisfied.

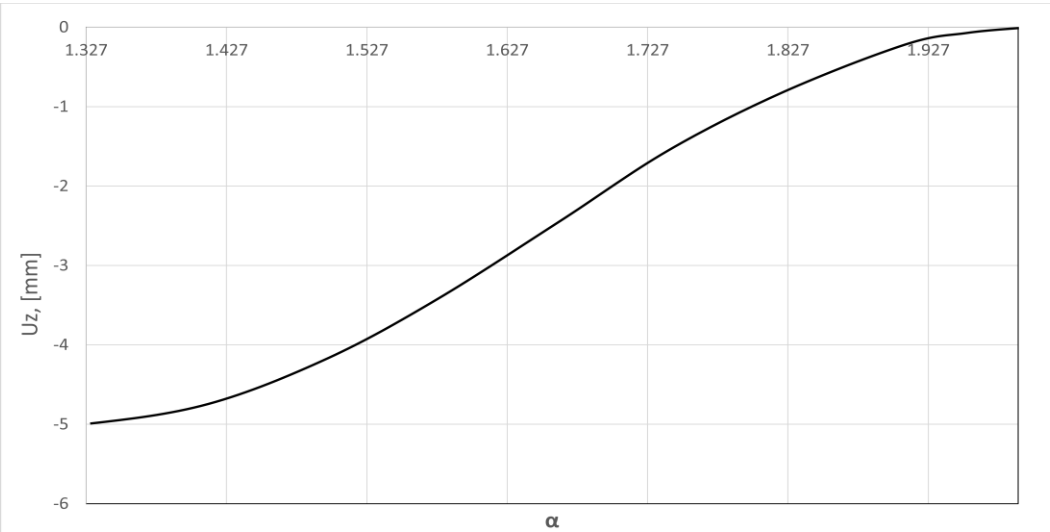

Figure 2. Diagram of normal displacements u_z at the draft of the curvilinear support

Bending moments are shown in Fig. 3.

The results shown in Fig. 2, 3 correlate to results obtained by Krivoshapko S.N. [11] for the developable helicoid in the form of Eq.1.

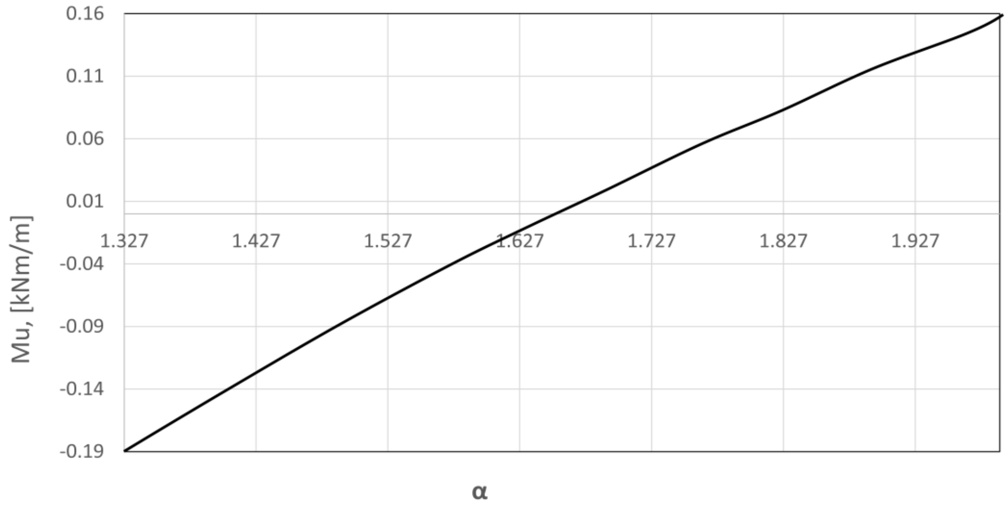

Figure 3. Diagram of bending moments M_u at the draft of the curvilinear support

5. Conclusion

There is the investigation on the analytical approach to calculation of a developable helicoid curvilinear support draft which is based on the three ordinary differential equations in dimensionless displacements U, V, W solved by asymptotic method of small parameter for the developable helicoid determined through the axes u and s, and a radius of a helical evolvent a_0 (Eq. 2).
Numerical experiments are conducted. The results for normal displacements and bending moments obtained by this method are shown and they correlate to the engineering practice and numerical results obtained by similar analytical method applied to the developable helicoid determined through the axes u and v, and a radius of the cylinder a (Eq. 1).
Despite the fact that the task is solved in “pseudo-moments”, they can be easily transverse into traditional engineering bending moments by the compact formula.
It is shown that the analytical methods can also be successfully used for calculation of the complex form surfaces. As a future research, it would be interesting to solve the task using more members of series, as well as the task for developable helicoid with other boundary conditions.

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Acknowledgments
This paper was financially supported by the Ministry of Education and Science of the Russian Federation on the program to improve the competitiveness of Peoples' Friendship University of Russia (RUDN University) among the world's leading research and education centres in the 2016-2020. This publication was prepared with the support of the "RUDN University Program 5-100".