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An Efficient Method of Finding Stress Solutions in Porous Material under Axial Symmetry

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Abstract. In the mechanics of porous and powder materials the system of equations comprising the pyramid yield criterion together with the stress equilibrium equations under conditions of axial symmetry forms a statically determinate system at edge regimes. The results presented here for this system are consequently independent of any flow rule that may be chosen to calculate the deformation and also independent of whether elastic strains are included. The stress equilibrium equations are written relative to a coordinate system in which the coordinate curves coincide with the trajectories of the principal stress directions. Then, a method of finding the trajectories of the principal stresses is developed.

1. Introduction

In the case of several yield criteria widely used in applications, the system of equations comprising the yield criterion together with the stress equilibrium equations forms a statically determinate system under certain conditions. This system can be studied without using any flow rule that may be chosen to calculate the deformation and independently of whether elastic strains are included. In particular, much work has been carried out for plane strain deformation. The geometric properties of the principal line coordinate system (i.e. the coordinate curves of this coordinate system coincide with trajectories of the principal stress directions) for the Tresca yield criterion have been derived in [1]. This result has been extended to the Mohr-Coulomb yield criterion in [2]. Several methods are available for calculating the field of stress for the Tresca yield criterion. In particular, the R – S method has been developed in [3]. Another widely used method is based on Mikhlin's variables [4, 5]. The R – S method has been extended to the Mohr-Coulomb yield criterion in [6]. The pyramid yield criterion for porous and powder materials has been proposed in [7]. The R – S method and the method based on Mikhlin's variables have been developed for this yield criterion in [8] and [9], respectively. In the present paper, a new method for determining the field of stress under axial symmetry is proposed for the pyramid yield criterion. The development of this method requires the generalization of the results reported in [1, 2] on axisymmetric deformation of materials obeying the pyramid yield criterion.

2. Material Model

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The piece-wise linear yield criterion in terms of the principal stresses σ_1, σ_2 and σ_3 proposed in [7] reads

$$\frac{\left|\sigma_i-\sigma_j\right|}{2\tau_s}+\frac{|\sigma|}{p_s}=1 \tag{1}$$

where $i,\ j=1,2,3$, $i\neq j$ and

$$\sigma=\frac{\sigma_1+\sigma_2+\sigma_3}{3} \tag{2}$$

is the hydrostatic stress. Also, τ_s is the shear yield stress and p_s is the yield stress in hydrostatic compression. Both τ_s and p_s depend of on the porosity, υ. In particular, $p_s\to\infty$ as $\upsilon\to 0$. In this case, the yield criterion (1) approaches the Tresca yield criterion. In what follows, it is assumed that the porosity is uniformly distributed. Therefore, τ_s and p_s are independent of space coordinates. In the case of axisymmetric deformation the circumferential stress σ_θ is one of the principal stresses. Without loss of generality, it is possible to assume that

$$\sigma_\theta\equiv\sigma_3 \quad \text{and} \quad \sigma_1>\sigma_2 \tag{3}$$

In general, under conditions of axial symmetry it is necessary to consider several edge and face regimes of the yield criterion (1). For practical applications, the edge regimes at which $\sigma_1\geq\sigma_\theta\geq\sigma_2$ are most important. There are four regimes of this kind. Using (1) these regimes can be represented as

$$\frac{\sigma_1-\sigma_2}{2\tau_s}-\frac{\sigma}{p_s}=1,\quad \frac{\sigma_1-\sigma_\theta}{2\tau_s}-\frac{\sigma}{p_s}=1 \tag{4}$$

in the case of regime 1,

$$\frac{\sigma_1-\sigma_2}{2\tau_s}-\frac{\sigma}{p_s}=1,\quad \frac{\sigma_\theta-\sigma_2}{2\tau_s}-\frac{\sigma}{p_s}=1 \tag{5}$$

in the case of regime 2,

$$\frac{\sigma_1-\sigma_2}{2\tau_s}+\frac{\sigma}{p_s}=1,\quad \frac{\sigma_1-\sigma_\theta}{2\tau_s}+\frac{\sigma}{p_s}=1 \tag{6}$$

in the case of regime 3, and

$$\frac{\sigma_1-\sigma_2}{2\tau_s}+\frac{\sigma}{p_s}=1,\quad \frac{\sigma_\theta-\sigma_2}{2\tau_s}+\frac{\sigma}{p_s}=1 \tag{7}$$

in the case of regime 4. In the case of regimes 1 and 2 the hydrostatic stress can vary in the range $0\geq\sigma\geq-p_s$ and in the case of regimes 3 and 4 in the range $0\leq\sigma\leq p_s$. In the case of regimes 1 and 3

$$\sigma_2=\sigma_\theta \tag{8}$$

and in the case of regimes 2 and 4

$$\sigma_1 = \sigma_\theta \tag{9}$$

The velocity equations follow from the associated flow rule. These equations are given in [7]. However, they are not involved in the subsequent analysis.

3. A Geometric Property of Principal Stress Trajectories

The yield criterion formulated in the previous section should be complemented with the equilibrium equations. It is convenient to introduce two coordinate systems, a cylindrical coordinate system (r, θ, z) whose *z*- axis coincides with the axis of symmetry of the boundary value problem such that its solution is independent of θ and a principal line coordinate system (ξ, θ, η). The coordinate curves of the principal line coordinate system coincide with the trajectories of the principal stresses. Without loss of generality it is assumed that the $\xi -$curves coincide with the trajectories of the stress σ_1 and the $\eta -$curves with the trajectories of the stress σ_2. Since the shear stresses vanish in the principal line coordinate system, the equilibrium equations referred to this system take the form [10]

$$\frac{\partial(rh_\eta\sigma_1)}{\partial\xi} - \sigma_\theta h_\eta \frac{\partial r}{\partial\xi} - \sigma_2 r\frac{\partial h_\eta}{\partial\xi} = 0, \quad \frac{\partial(rh_\xi\sigma_2)}{\partial\eta} - \sigma_1 r\frac{\partial h_\xi}{\partial\eta} - \sigma_\theta h_\xi \frac{\partial r}{\partial\eta} = 0. \tag{10}$$

In the case of regimes 1 and 3 equations (8) and (10) combine to give

$$(\sigma_1 - \sigma_2)\left(h_\eta\frac{\partial r}{\partial\xi} + r\frac{\partial h_\eta}{\partial\xi}\right) + rh_\eta\frac{\partial\sigma_1}{\partial\xi} = 0, \quad -(\sigma_1 - \sigma_2)\frac{\partial h_\xi}{\partial\eta} + h_\xi\frac{\partial\sigma_2}{\partial\eta} = 0. \tag{11}$$

In the case of regimes 2 and 4 equations (9) and (10) combine to give

$$-(\sigma_1 - \sigma_2)\left(h_\xi\frac{\partial r}{\partial\eta} + r\frac{\partial h_\xi}{\partial\eta}\right) + rh_\xi\frac{\partial\sigma_2}{\partial\eta} = 0, \quad (\sigma_1 - \sigma_2)\frac{\partial h_\eta}{\partial\xi} + h_\eta\frac{\partial\sigma_1}{\partial\xi} = 0. \tag{12}$$

Eliminating the difference $\sigma_1 - \sigma_2$ in (11) by means of (4) and (6) yields

$$2\tau_s\left(1 \pm \frac{\sigma}{p_s}\right)\left(h_\eta\frac{\partial r}{\partial\xi} + r\frac{\partial h_\eta}{\partial\xi}\right) + rh_\eta\frac{\partial\sigma_1}{\partial\xi} = 0, \quad -2\tau_s\left(1 \pm \frac{\sigma}{p_s}\right)\frac{\partial h_\xi}{\partial\eta} + h_\xi\frac{\partial\sigma_2}{\partial\eta} = 0 \tag{13}$$

where the upper sign corresponds to regime 1 and the lower sign to regime 3. Eliminating the difference $\sigma_1 - \sigma_2$ in (12) by means of (5) and (7) yields

$$-2\tau_s\left(1 \pm \frac{\sigma}{p_s}\right)\left(h_\xi\frac{\partial r}{\partial\eta} + r\frac{\partial h_\xi}{\partial\eta}\right) + rh_\xi\frac{\partial\sigma_2}{\partial\eta} = 0, \quad 2\tau_s\left(1 \pm \frac{\sigma}{p_s}\right)\frac{\partial h_\eta}{\partial\xi} + h_\eta\frac{\partial\sigma_1}{\partial\xi} = 0. \tag{14}$$

where the upper sign corresponds to regime 2 and the lower sign to regime 4. It follows from (2), (8) and (9) that $3\sigma = \sigma_1 + 2\sigma_2$ in the case of regimes 1 and 3 and $3\sigma = 2\sigma_1 + \sigma_2$ in the case of regimes 2 and 4. Using these equations together with equations (4) to (7) it is possible to express σ_1 and σ_2 in terms of σ. As a result,

$$\sigma_1 = (1+2k_s)\sigma + 4\tau_s/3, \quad \sigma_2 = (1-k_s)\sigma - 2\tau_s/3 \tag{15}$$

in the case of regime 1,

$$\sigma_1 = (1+k_s)\sigma + 2\tau_s/3, \quad \sigma_2 = (1-2k_s)\sigma - 4\tau_s/3 \tag{16}$$

in the case of regime 2,

$$\sigma_1 = (1-2k_s)\sigma + 4\tau_s/3, \quad \sigma_2 = (1+k_s)\sigma - 2\tau_s/3 \tag{17}$$

in the case of regime 3, and

$$\sigma_1 = (1-k_s)\sigma + 2\tau_s/3, \quad \sigma_2 = (1+2k_s)\sigma - 4\tau_s/3 \tag{18}$$

in the case of regime 4. In equations (15) to (18), $k_s = 2\tau_s/(3p_s)$.
In the case of regime 1, eliminating σ_1 and σ_2 in (13) by means of (15) yields

$$\frac{\partial \ln(rh_\eta)}{\partial \xi} + \frac{(1+2k_s)}{2\tau_s(1+\sigma/p_s)}\frac{\partial \sigma}{\partial \xi} = 0, \quad \frac{\partial \ln h_\xi}{\partial \eta} - \frac{(1-k_s)}{2\tau_s(1+\sigma/p_s)}\frac{\partial \sigma}{\partial \eta} = 0 . \tag{19}$$

Each of these equations can be immediately integrated to give

$$\frac{3k_s}{(1+2k_s)}\ln(rh_\eta) + \ln\left(1+\frac{\sigma}{p_s}\right) = \ln\left[C_1(\eta)\right], \quad \frac{3k_s}{(1-k_s)}\ln h_\xi - \ln\left(1+\frac{\sigma}{p_s}\right) = \ln\left[C_2(\xi)\right]. \tag{20}$$

Here $C_1(\eta)$ is an arbitrary function of only η and $C_2(\xi)$ is an arbitrary function of only ξ. Eliminating σ between the equations in (20) yields

$$\frac{3k_s}{(1+2k_s)}\ln(rh_\eta) + \frac{3k_s}{(1-k_s)}\ln h_\xi = \ln\left[C_1(\eta)C_2(\xi)\right] . \tag{21}$$

It is seen from this equation that different choices of the functions $C_1(\eta)$ and $C_2(\xi)$ merely change the scale of the $\eta-$ and $\xi-$ curves, respectively. Therefore, it is always possible to choose $C_1(\eta) = C_2(\xi) = 1$. Then, equation (21) becomes

$$rh_\eta h_\xi^{t_1} = 1, \quad t_1 = \frac{1+2k_s}{1-k_s}. \tag{22}$$

In this case equation (20) supplies

$$\frac{\sigma}{p_s} = h_\xi^{t_0} - 1, \quad t_0 = \frac{3k_s}{1-k_s}. \tag{23}$$

Using equations (13) to (18) regimes 2, 3 and 4 can be treated in a similar manner.

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4. Mapping between the Cylindrical and Principal Line Coordinate Systems

The cylindrical and principal line coordinate systems are shown in Fig. 1 where ψ is the orientation of the $\xi-$curves relative to the $r-$axis.

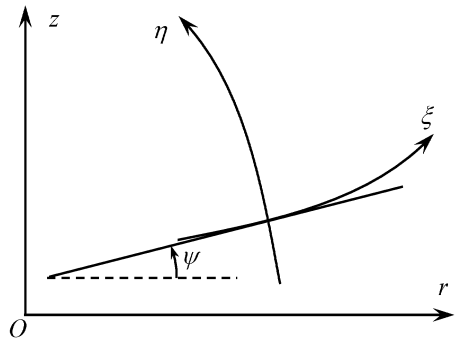

Figure 1. Cylindrical (*r, z*) and principal line (ξ, η) coordinate systems.

It follows from the geometry of this figure that

$$\frac{\partial r}{\partial \xi} = h_\xi \cos\psi, \quad \frac{\partial r}{\partial \eta} = -h_\eta \sin\psi, \quad \frac{\partial z}{\partial \xi} = h_\xi \sin\psi, \quad \frac{\partial z}{\partial \eta} = h_\eta \cos\psi. \tag{24}$$

If regime1 is operative then this equation and (22) result in

$$\frac{\partial r}{\partial \xi} = h\cos\psi, \quad \frac{\partial r}{\partial \eta} = -\frac{\sin\psi}{rh^{t_1}}, \quad \frac{\partial z}{\partial \xi} = h\sin\psi, \quad \frac{\partial z}{\partial \eta} = \frac{\cos\psi}{rh^{t_1}} \tag{25}$$

where $h \equiv h_\xi$. The compatibility equations are

$$\frac{\partial^2 r}{\partial \xi \partial \eta} = \frac{\partial^2 r}{\partial \eta \partial \xi}, \quad \frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{\partial^2 z}{\partial \eta \partial \xi}. \tag{26}$$

Substituting (25) into (26) gives

$$\begin{aligned}
&\cos\psi \frac{\partial h}{\partial \eta} - h\sin\psi \frac{\partial \psi}{\partial \eta} + \frac{\cos\psi}{rh^{t_1}}\frac{\partial \psi}{\partial \xi} - \frac{t_1 \sin\psi}{rh^{t_1+1}}\frac{\partial h}{\partial \xi} = \frac{\sin\psi\cos\psi}{r^2 h^{t_1-1}}, \\
&\sin\psi \frac{\partial h}{\partial \eta} + h\cos\psi \frac{\partial \psi}{\partial \eta} + \frac{\sin\psi}{rh^{t_1}}\frac{\partial \psi}{\partial \xi} + \frac{t_1 \cos\psi}{rh^{t_1+1}}\frac{\partial h}{\partial \xi} = -\frac{\cos^2\psi}{r^2 h^{t_1-1}}.
\end{aligned} \tag{27}$$

Multiplying the first of (27) by $\sin\psi$ and subtracting from the result the second of (27) multiplied by $\cos\psi$ yields

$$\frac{\partial \psi}{\partial \eta} + \frac{t_1}{rh^{t_1+2}}\frac{\partial h}{\partial \xi} = -\frac{\cos\psi}{r^2 h^{t_1}}. \tag{28}$$

In a similar manner, if the first and second of (27) are multiplied by $\cos\psi$ and $\sin\psi$ respectively, and the sum of the resulting equations is taken, it is found that

$$rh^{t_1}\frac{\partial h}{\partial \eta}+\frac{\partial \psi}{\partial \xi}=0. \tag{29}$$

Using a standard procedure it is possible to show that (28) and (29) is a hyperbolic system of equations for *h* and ψ with characteristic curves (termed $\alpha-$ and $\beta-$ lines) given by

$$\frac{d\xi}{d\eta}=\pm\frac{\sqrt{t_1}}{rh^{t_1+1}}. \tag{30}$$

where the lower and upper signs refer to the $\alpha-$lines and $\beta-$lines respectively. The characteristic relations are

$$\pm d\psi+\frac{\sqrt{t_1}}{h}dh=-\frac{h\cos\psi}{r\sqrt{t_1}}d\xi. \tag{31}$$

As before, the lower and upper signs correspond to the $\alpha-$lines and $\beta-$lines respectively. Equations (30) and (31) should be solved together with equation (25) for *r*. This system of equations is similar to that that describes axisymmetric flow of Tresca's solids [11]. Numerical techniques for solving the latter are well documented [11]. These techniques can be adopted for solving the system of equations derived. Once this system has been solved, equation (23) supplies the distribution of σ and equation (15) the distributions of σ_1 and σ_2. Since the angle ψ has been found, the distribution of the stress components in the cylindrical coordinate system follows from the standard transformation equations for stress components.
Regimes 2, 3 and 4 can be treated in a similar manner.

5. Conclusions

On the assumption of axial symmetry conditions the system of equations comprising the pyramid yield criterion and the equilibrium equations has been studied in a curvilinear orthogonal coordinate system in which the coordinate curves coincide with trajectories of the principal stress directions. It has been shown that the scale factors of the coordinate curves should satisfy equation (22) in the case of regime 1 or similar equations in the case of the other edge regimes of the pyramid yield criterion. Using this equation a method of finding stress solutions has been developed. In particular, one of the scale factors of the principal line coordinate system and the orientation of the trajectory of the major principal stress should satisfy equations (28) and (29). These equations form a hyperbolic system with the characteristic curves given by (30) and the characteristic relations given by (31). Numerical techniques for solving this type of equations are available in the literature [11].

6. References

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