

Properties of canonical transformations of linear Hamiltonian systems

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Abstract. In this paper we consider the linear Hamiltonian systems of differential equations. The Hamiltonian systems have an important role in fluid mechanics and in statistical mechanics. Firstly, we prove some properties of canonical transformations of linear Hamiltonian systems. Further, we get the new method to find the generating function of a canonical transformation. We get a system of matrix equations for finding the generating function of a canonical transformation. In various cases we obtain the solution of the system. Then we use the system for normalization the Hamiltonian matrix. We apply the system of matrix equations to transform the Hamiltonian matrix from one normal form to another. Further, we get the solution of the matrix Riccati equation. This nonlinear equation has an important role in optimal control problems, multivariable and large-scale systems, scattering theory, estimation, detection and transportation. Finally, we transform the Hamiltonian from the complex form to the real form. An illustrative example for the proposed method is given. Therefore, we obtain the new method of normalization of the quadratic Hamiltonian. With this method we can investigate the stability of the solution of the Hamiltonian systems.

1. Introduction

The Hamiltonian systems are of potential interest for statistical mechanics and for fluid mechanics. The Hamiltonian systems arise frequently in the design and construction of tunnels and in the problem of filtration. It occurs in the production of construction materials. In the design of underground and hydraulic structures one solves this problem. We consider the canonical system of differential equations:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

with a Hamiltonian of the type:

$$H(x, y) = 1/2 x^T A x + x^T B y + 1/2 y^T C y,$$

where both x and y are n -dimensional column-vectors of conjugate Hamiltonian variables; A , B , C are real n -by- n matrices, A and C are symmetric matrices. The Hamiltonian matrix corresponding to this system has the form:

$$V = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix}. \quad (1)$$



In the paper [1] the author proves the possibility of normalization of Hamiltonian with nondegenerate corresponding Hamiltonian matrix. In the paper [2] the author considers the case of nonzero eigenvalues of Hamiltonian matrix. In the works [3-5] the authors obtain other methods of normalization of Hamiltonian. The aim of this paper is to find the generating function of a canonical transformation. One of the most intensely studied matrix nonlinear equations arising in engineering is the Riccati equation [5 - 8]. This equation applies in optimal control problems, scattering theory, estimation, detection, transportation, multivariable and large-scale systems. We get the solution of this equation.

2. System of matrix equations

Let $\mathbf{q}, \mathbf{p}$ be n -dimensional column-vectors of the new Hamiltonian variables. We make the canonical transformation with the help of the generating function:

$$S(\mathbf{x}, \mathbf{p}) = 1/2 \mathbf{p}^T K \mathbf{p} + \mathbf{p}^T L \mathbf{x} + 1/2 \mathbf{x}^T M \mathbf{x}, \quad (2)$$

where K, L, M are n -by- n matrices, K and M are symmetric, and L is a non-singular matrix. Then from the equations:

$$\frac{\partial S}{\partial \mathbf{p}} = \mathbf{q}, \quad \frac{\partial S}{\partial \mathbf{x}} = \mathbf{y}$$

we obtain an expression of the old variables through the new ones:

$$\begin{aligned} \mathbf{x} &= L^{-1} \mathbf{q} - L^{-1} K \mathbf{p}; \\ \mathbf{y} &= M L^{-1} \mathbf{q} + (L^T - M L^{-1} K) \mathbf{p} \end{aligned} \quad (3)$$

We make the necessary calculations and obtain a new Hamiltonian:

$$\begin{aligned} H(\mathbf{q}, \mathbf{p}) &= \frac{1}{2} \mathbf{q}^T (L^T)^{-1} (MCM + 2BM + A) L^{-1} \mathbf{q} + \\ &+ \mathbf{q}^T \left[(L^T)^{-1} (B + MC) L^T - (L^T)^{-1} (MCM + MB^T + BM + A) L^{-1} K \right] \mathbf{p} + \\ &+ \frac{1}{2} \mathbf{p}^T \left[K (L^T)^{-1} (MCM + 2BM + A) L^{-1} K - K (L^T)^{-1} (MC + B) L^T - \right. \\ &\left. - L(CM + B^T) L^{-1} K + LCL^T \right] \mathbf{p} = \frac{1}{2} \mathbf{q}^T A_0 \mathbf{q} + \mathbf{q}^T B_0 \mathbf{p} + \frac{1}{2} \mathbf{p}^T C_0 \mathbf{p}, \end{aligned}$$

where

$$\begin{aligned} A_0 &= (L^T)^{-1} (MCM + MB^T + BM + A) L^{-1}; \\ B_0 &= (L^T)^{-1} (B + MC) L^T - A_0 K; \\ C_0 &= -KA_0 K - KB_0 - B_0^T K + LCL^T; \quad A_0 = A_0^T; \quad C_0 = C_0^T \end{aligned}$$

Here the following obvious identity was used:

$$2\mathbf{q}^T B M \mathbf{q} = \mathbf{q}^T (BM + MB^T) \mathbf{q}$$

Thus, if we want to bring the canonical system of differential equations with the matrix (1) by the canonical transformation (3) to a system with a Hamiltonian matrix

$$V_0 = \begin{pmatrix} B_0^T & C_0 \\ -A_0 & -B_0 \end{pmatrix}, \quad C_0 = C_0^T, \quad A_0 = A_0^T, \quad (4)$$

we seek the unknown matrices K, L, M out of the following system of matrix equations:

$$\begin{aligned} MCM + MB^T + BM + A &= L^T A_0 L, \\ L(B^T + CM)L^{-1} &= KA_0 + B_0^T, \\ KA_0 K + KB_0 + B_0^T K + C_0 &= LCL^T \end{aligned} \quad (5)$$

In this system, we seek the unknown matrices M and K as symmetric, and the matrix L as a non-singular. As a result, the following conclusion is proved.

Theorem 1. The canonical transformation given by the generating function (2) with the non-singular matrix L leads the Hamiltonian matrix (1) to the form (4) if and only if the matrices K, L, M of the generating function are solutions of the system (5).

3. Solution of the system of matrix equations

With the help of system (5) it is possible to simplify the initial Hamiltonian matrix, to lead it to some normal form, and to pass from one normal form to another. Let us consider various cases.

Case 1. Consider the generating function (2) with the matrices $M = O$, $K = O$. Then the system (5) takes on form:

$$\begin{aligned} A &= L^T A_0 L, \\ LB^T L^{-1} &= B_0^T, \\ C_0 &= LCL^T \end{aligned}$$

Further we can conclude the following. With the help of a transformation given by this generating function, we can reduce the submatrix A or C of the Hamiltonian matrix (1) to diagonality. We also can reduce the submatrix B to the Jordan normal form. If the submatrix C is positive definite, then we can simultaneously bring the submatrix C to the identity matrix, and the submatrix A to the diagonal matrix.

Case 2. We consider the Hamiltonian matrix (1) with eigenvalues $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_n$. We want to reduce the matrix to the following normal form:

$$\begin{aligned} \Phi &= \begin{pmatrix} U & I \\ O & -U^T \end{pmatrix}, \\ U &= \begin{pmatrix} \lambda_1 & \varepsilon_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \varepsilon_{n-1} \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}, \\ I &= \begin{pmatrix} \varepsilon_n & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_{n+1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \varepsilon_{2n-1} \end{pmatrix}, \\ \varepsilon_i &= \begin{cases} 0, 1; i = \overline{1, n-1} \\ 0, \pm 1; i = \overline{n, 2n-1} \end{cases} \end{aligned} \quad (6)$$

Then the system of matrix equations (5) takes on form:

$$\begin{aligned} MCM + MB^T + BM + A &= O, \\ L(B^T + CM)L^{-1} &= U, \\ KU^T + UK + I &= LCL^T \end{aligned} \quad (7)$$

The first equation of the resulting system is called the Riccati matrix equation [5 - 7]. Unknown matrices are found sequentially out of this system. We shall find the symmetric matrix M out of the first equation. We shall find the matrix L out of the second equation, which transforms the matrix $B^T + CM$ to the normal Jordan form. We shall find the symmetric matrix

K out of the third equation. In paper [4] methods of system solution are considered in case when the matrix C is positive definite, and the matrix $I = O$.

Theorem 2. The system of matrix equations (7) has a solution if and only if there exists a symplectic matrix

$$T = \begin{pmatrix} F & H \\ G & W \end{pmatrix}, \quad T^{-1}VT = \Phi, \quad \det F \neq 0, \quad (8)$$

with square submatrices F, G, H, W .

Proof. Let the system (7) have a solution. Then the transformation matrix (3) is symplectic, and its submatrix in the upper left corner is non-singular.

We now consider the symplectic matrix (8). The matrix (8) coincides with the transformation matrix (3) if

$$\begin{aligned} L^{-1} &= F, \quad -L^{-1}K = H, \\ ML^{-1} &= G, \quad L^T - ML^{-1}K = W \end{aligned}$$

We find out of the first three equations of this system:

$$\begin{aligned} L &= F^{-1}, \\ K &= -F^{-1}H, \\ M &= GF^{-1} \end{aligned} \quad (9)$$

The last equation is satisfied identically if we apply the matrices L, K, M to it and use the properties of the symplectic matrix [1 - 2]:

$$\begin{aligned} F^T G &= G^T F, \\ H^T W &= W^T H, \\ F^T W - G^T H &= E \end{aligned}$$

Real:

$$\begin{aligned} L^T - ML^{-1}K &= (F^T)^{-1} + GF^{-1}H = \\ &= (F^T)^{-1} + (F^T)^{-1}G^T H = W \end{aligned}$$

It follows from the first property that the matrix M is symmetric. Thus, the matrices (9) are a solution of the system (7).

Theorem 3. Suppose that there exists a solution of the first equation of system (7). Then there exists a symmetric matrix I and a Jordan normal matrix U for whereby the system (7) has a solution.

Proof. Suppose $M = M^T$ is a solution of the first equation of system (7). Then the matrix L reduces the matrix $CM + B^T$ to the Jordan normal form U . The matrix I ensures the solvability of the third equation of the system. For example, for $I = LCL^T$, the third equation of system (7) has a solution $K=O$.

Various methods for solving the matrix Riccati equation are considered in papers [5 - 7].

Case 3. Suppose that the non-zero eigenvalues of the Hamiltonian matrix (1) are purely imaginary ($\lambda_j = i\omega_j, j = \overline{1, k}$) and do not form Jordan blocks of higher than the first order. Then it can be reduced to the normal form (6), where

$$\begin{aligned} U &= U_k \oplus O_{n-k}; \quad I = O_k \oplus I_{n-k}; \\ U_k &= \begin{pmatrix} i\omega_1 & 0 & 0 & \dots & 0 \\ 0 & i\omega_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & i\omega_k \end{pmatrix}, \end{aligned}$$

$$I_{n-k} = \begin{pmatrix} \varepsilon_1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \varepsilon_{n-k} \end{pmatrix},$$

$$\varepsilon_i = 0; 1; -1, i = \overline{1, n-k}.$$

The normal form is a complex matrix. Let us proceed to the next real form:

$$\begin{pmatrix} O & C \\ -A & O \end{pmatrix},$$

$$C = -iU_k \oplus I_{n-k}, \quad A = -iU_k \oplus O_{n-k}$$

The system of matrix equations (5) takes on form:

$$MIM + MU + UM = L^T AL,$$

$$L(U + IM)L^{-1} = KA,$$

$$KAK + C = LIL^T$$

It is easy to verify that the following matrices satisfy the resulting system:

$$M = -\frac{1}{2}iE_k \oplus O_{n-k}, \quad L = E, \quad K = iE_k \oplus O_{n-k}$$

As a result, we obtain the following canonical transformation:

$$x_1 = q_1 - ip_1, \quad x_2 = q_2 - ip_2, \quad \dots, \quad x_k = q_k - ip_k, \quad x_{k+1} = q_{k+1}, \quad \dots, \quad x_n = q_n,$$

$$y_1 = -\frac{1}{2}iq_1 + \frac{1}{2}p_1, \quad y_2 = -\frac{1}{2}iq_2 + \frac{1}{2}p_2, \quad \dots, \quad y_k = -\frac{1}{2}iq_k + \frac{1}{2}p_k,$$

$$y_{k+1} = p_{k+1}, \quad \dots, \quad y_n = p_n$$

The new Hamiltonian has the form:

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{j=1}^k \omega_j (p_j^2 + q_j^2) + \frac{1}{2} \sum_{j=k+1}^n \varepsilon_{j-k} p_j^2$$

Example. We consider the Hamiltonian matrix (1) with submatrices:

$$A = \begin{pmatrix} 1 & 8 \\ 8 & 13 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad C = E$$

Let us find the eigenvalues of the matrix V . We obtain

$$\lambda_1 = 2, \quad \lambda_2 = -2, \quad \lambda_3 = \lambda_4 = 0$$

The rank of the matrix V is 3. Therefore, the zero eigenvalues form a Jordan block of order 2. We shall reduce the matrix V to the normal form (6) with submatrices.

$$U = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to verify that the following matrices are system solution (7):

$$M = \begin{pmatrix} 1 & -2 \\ -2 & -3 \end{pmatrix}, \quad L = E, \quad K = \begin{pmatrix} 1/4 & 0 \\ 0 & 0 \end{pmatrix}$$

We obtain the canonical transformation of the form (3). The new Hamiltonian has the following form:

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{q}^T U \mathbf{p} + \frac{1}{2} \mathbf{p}^T I \mathbf{p} = 2p_1 q_1 + \frac{1}{2} p_2^2$$

4. Conclusions

We obtain the new convenient practical method to find the generating function of a canonical transformation. We use the method for normalization of the quadratic Hamiltonian. With this method we can investigate the stability of underground structures [8 - 12]. We get the solution of the matrix Riccati equation. This nonlinear matrix equation has an important role in optimal control problems [9 - 13], multivariable and large-scale systems, scattering theory, estimation, detection and transportation [14 – 22].

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