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Plates and beams of asymmetric structure in thickness

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Abstract. The theory of plates and beams of asymmetric structure in thickness is constructed. It is shown that, in general, the complete problem for this kind of a structure is not divided into a plane problem and a bending problem. Here stress-strain state of plates and beams of asymmetric structure in thickness is analyzed by the mathematical method without using any simplifying assumptions. Simple theories were obtained for practical applications by an asymptotic method. As examples, the dynamic problems for a two-layer beam and a plate with a gradient of properties in thickness were calculated.

1. Introduction

Plates and beams are widely used in constructions. To ensure that they work reliably, they must be calculated correctly. Often they have an asymmetrical structure in thickness. Such structures do not have a middle plane, which is a plane of symmetry in terms of its geometric and physical parameters. Examples of such constructions are asymmetric layered structures, structures with a gradient of properties in thickness, and asymmetric constructions with variable thickness. It is known that the complete problem for a symmetrical beam or plate is divided into two problems. They are a plane problem and a bending problem. A different situation occurs for asymmetric plates and beams. In this case the complete problem, generally speaking, is not broken up into a plane problem and a bending problem. So the elasticity relations for forces and moments simultaneously contain both tangential and flexural deformations. In addition, the inertia of rotation must be taken into account in the equations of motion.

Many papers are devoted to arbitrary laminated thin-walled structures and structures with gradient of properties in thickness [1], [2]. In most of them, either numerical methods or models based on some hypotheses are used.

2. Formulation of the problem and initial equations

First we obtain two-dimensional (2D) equations for the plate. The cross-section of the plate related to the Cartesian coordinate system is shown in figure 1.

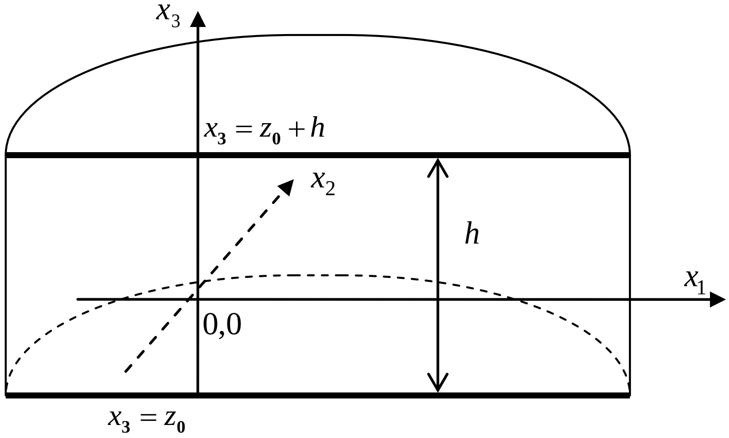

Figure 1. The cross section of the plate with gradient physical properties

As the initial equations, we take the three-dimensional (3D) elasticity equations.
They are written as follows
Equations of motion

$$\frac{\partial\sigma_i}{\partial x_i}+\frac{\partial\sigma_{ij}}{\partial x_j}+\frac{\partial\sigma_{i3}}{\partial x_3}=\rho\frac{\partial^2 u_i}{\partial t^2},\quad \frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}+\frac{\partial\sigma_3}{\partial x_3}=\rho\frac{\partial^2 u_3}{\partial t^2}$$

Strain-displacement formulas

$$e_{ii}=\frac{\partial u_i}{\partial x_i},\quad e_{ij}=\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j},\quad e_{i3}=\frac{\partial u_i}{\partial x_3}+\frac{\partial u_3}{\partial x_i},\quad e_{33}=\frac{\partial u_3}{\partial x_3}$$

Hooke's law, recorded in a form convenient for the future, resolved relative to the main stresses

$$\sigma_{ii}=\frac{E}{\mathbf{1}-\nu^2}(e_{ii}+\nu e_{jj})+\left[\frac{\nu}{\mathbf{1}-\nu}\sigma_{33}\right],\ \sigma_{ij}=\frac{E}{\mathbf{2}(\mathbf{1}+\nu)}e_{ij},\ e_{i3}=\frac{\mathbf{1}+\nu}{E}\sigma_{i3},\ \sigma_{33}=Ee_{33}+\nu(\sigma_{11}+\sigma_{22})$$

Here σ_{nm} are the components of the stress tensor, e_{nm} are the components of the strain tensor, E is the modulus of elasticity, ν is the Poisson ratio.
As a result of the asymptotic analysis of 3D equations, which for displacements and deformations coincides with that performed in [3], with an accuracy up to values of the order ε

$$\varepsilon = O(\eta^1+\eta^{2-2s}) \tag{1}$$

(s is the variability of stress – strain state with respect to the coordinates x_1 and x_2) we obtain the following expansions for displacements and deformations with coordinate x_3

$$v_i=u_i(x_3=z_0),\quad w=-u_3(x_3=z_0),\quad u_i=v_i-x_3\gamma,\quad e_{ii}=\varepsilon_i+x_3\kappa,\quad e_{12}=\omega+\mathbf{2}x_3\tau$$

$$\varepsilon_i=\frac{\partial v_i}{\partial x_i},\quad \omega=\frac{\partial v_i}{\partial x_i},\quad \gamma_i=-\frac{\partial w}{\partial x_i},\quad \kappa_i=-\frac{\partial\gamma_i}{\partial x_i},\quad \tau=-\frac{\partial\gamma_1}{\partial x_2}-\frac{\partial\gamma_2}{\partial x_1} \tag{2}$$

Integrating the stresses to the coordinate x_3, we obtain the forces and moments

$$T_i=\int_{z_0}^{z_0+h}\sigma_{ii}\,\mathrm{d}x_3,\quad S=\int_{z_0}^{z_0+h}\sigma_{12}\,\mathrm{d}x_3,\quad N_i=-\int_{z_0}^{z_0+h}\sigma_{i3}\,\mathrm{d}x_3,\quad H=\int_{z_0}^{z_0+h}\sigma_{12}x_3\,\mathrm{d}x_3,\quad G_i=-\int_{z_0}^{z_0+h}\sigma_{ii}x_3\,\mathrm{d}x_3 \tag{3}$$

Then taking into account last formulas (2), (3) we obtain the following elasticity relations for forces and moments

$$T_i = A_{11} h\varepsilon_i + A_{12} h\varepsilon_j + C_{11} h^2 \kappa_i + C_{12} h^2 \kappa_j, \quad S = Ah\omega + 2Ch^2\tau \tag{4}$$

$$G_i = -M_{11} h^3 \kappa_i - M_{12} h^3 \kappa_j - C_{11} h^2 \varepsilon_i - C_{12} h^2 \varepsilon_j, \quad H = 2Mh^3\tau + Ch^2\omega$$

In the formulas (4) for the plate with the gradient of properties we introduce the notations

$$A_{11} = \frac{1}{h}\int_{z_0}^{z_0+h} \frac{E}{1-\nu^2}\,\mathrm{d}\,x_3, \quad A_{12} = \frac{1}{h}\int_{z_0}^{z_0+h} \frac{\nu E}{1-\nu^2}\,\mathrm{d}\,x_3, \quad A = \frac{1}{h}\int_{z_0}^{z_0+h} \frac{E}{2(1+\nu)}\mathrm{d}\,x_3 \tag{5}$$

$$C = \frac{1}{h^2}\int_{z_0}^{z_0+h} \frac{E}{2(1+\nu)} x_3\,\mathrm{d}\,x_3, \quad C_{11} = \frac{1}{h^2}\int_{z_0}^{z_0+h} \frac{E}{1-\nu^2} x_3\,\mathrm{d}\,x_3, \quad C_{12} = \frac{1}{h^2}\int_{z_0}^{z_0+h} \frac{\nu E}{1-\nu^2} x_3\,\mathrm{d}\,x_3$$

$$M = \frac{1}{h^3}\int_{z_0}^{z_0+h} \frac{E}{2(1+\nu)} x_3^2\,\mathrm{d}\,x_3, \quad M_{11} = \frac{1}{h^3}\int_{z_0}^{z_0+h} \frac{E}{1-\nu^2} x_3^2\,\mathrm{d}x_3, \quad M_{12} = \frac{1}{h^3}\int_{z_0}^{z_0+h} \frac{\nu E}{1-\nu^2} x_3^2\,\mathrm{d}\,x_3$$

For a laminated plate consisting of homogeneous layers, constants $A_{11},\ldots, M_{12}$ can be written in the form

$$A_{11} = \frac{1}{h}\sum_{k=1}^{n} \frac{E_k h_k}{1-\nu_k{}^2}, \quad A_{12} = \frac{1}{h}\sum_{k=1}^{n} \frac{\nu_k E_k h_k}{1-\nu_k{}^2}, \quad A = \frac{1}{h}\sum_{k=1}^{n} \frac{E_k h_k}{2(1+\nu_k)} \tag{6}$$

$$C = \frac{1}{2h^2}\sum_{k=1}^{n} \frac{E_k (z_k{}^2 - z_{k-1}{}^2)}{2(1+\nu_k)}, \quad C_{11} = \frac{1}{2h^2}\sum_{k=1}^{n} \frac{E_k (z_k{}^2 - z_{k-1}{}^2)}{1-\nu_k{}^2}, \quad C_{12} = \frac{1}{2h^2}\sum_{k=1}^{n} \frac{\nu_k E_k (z_k{}^2 - z_{k-1}{}^2)}{1-\nu_k{}^2}$$

$$M = \frac{1}{3h^3}\sum_{k=1}^{n} \frac{E_k (z_k{}^3 - z_{k-1}{}^3)}{2(1+\nu_k)}, \quad M_{11} = \frac{1}{3h^3}\sum_{k=1}^{n} \frac{E_k (z_k{}^3 - z_{k-1}{}^3)}{1-\nu_k{}^2}, \quad M_{12} = \frac{1}{3h^3}\sum_{k=1}^{n} \frac{\nu_k E_k (z_k{}^3 - z_{k-1}{}^3)}{1-\nu_k{}^2}$$

$$h = \sum_{k=1}^{n} h_k$$

Here k is the number of the layer for which $z_{k-1} \le x_3 \le z_k$, h_k is the thickness of the k-th layer.

Integrating 3D equations of motion with respect to the coordinate x_3 with allowance for formulas (3), we obtain the equations of motion of the plate

$$\frac{\partial T_i}{\partial x_i} + \frac{\partial S}{\partial x_j} + X_i = h\rho_{(0)} \frac{\partial^2 v_i}{\partial t^2} - h^2 \rho_{(1)} \frac{\partial^2 \gamma_i}{\partial t^2}$$

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_2}{\partial x_2} + Z = h\rho_{(0)} \frac{\partial^2 w}{\partial t^2}, \quad N_i = \frac{\partial G_i}{\partial x_i} + \frac{\partial H}{\partial x_j} + h^2 \rho_{(1)} \frac{\partial^2 v_i}{\partial t^2} - h^3 \rho_{(2)} \frac{\partial^2 \gamma_i}{\partial t^2}$$

Formulas (7) and (8) refer to plates with a gradient of properties along the thickness and arbitrary laminated plates, respectively

$$\rho_{(0)} = \frac{1}{h}\int_{z_0}^{z_0+h} \rho\,\mathrm{d}\,x_3, \quad \rho_{(1)} = \frac{1}{h^2}\int_{z_0}^{z_0+h} \rho x_3\,\mathrm{d}\,x_3, \quad \rho_{(2)} = \frac{1}{h^3}\int_{z_0}^{z_0+h} \rho x_3{}^2\,\mathrm{d}\,x_3 \tag{7}$$

$$\rho_{(0)} = \frac{1}{h}\sum_{k=1}^{n} \rho_k h_k, \quad \rho_{(1)} = \frac{1}{2h^2}\sum_{k=1}^{n} \rho_k (z_k - z_{k-1})^2, \quad \rho_{(2)} = \frac{1}{3h^3}\sum_{k=1}^{n} \rho_k (z_k - z_{k-1})^3 \tag{8}$$

Using the equations of consistency of deformations

$$\frac{\partial^2 \varepsilon_2}{\partial x_1^2} + \frac{\partial^2 \varepsilon_1}{\partial x_2^2} - \frac{\partial^2 \omega}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 \kappa_2}{\partial x_1^2} + \frac{\partial^2 \kappa_1}{\partial x_2^2} - 2\frac{\partial^2 \tau}{\partial x_1 \partial x_2} = 0$$

strain - displacement formulas (2) and formulas

$$C_{12} + 2C = C_{11}, \quad M_{12} + 2M = M_{11}, \quad A_{12} + 2A = A_{11}$$

we write the equations of motion in the following form

$$A_{11} h \frac{\partial \varepsilon_i}{\partial x_i} + A_{12} h \frac{\partial \varepsilon_j}{\partial x_i} + Ah\frac{\partial \omega}{\partial x_j} + C_{11} h^2 \frac{\partial}{\partial x_i}(\kappa_i + \kappa_j) + X_i = h\rho_{(0)} \frac{\partial^2 v_i}{\partial t^2} - h^2 \rho_{(1)} \frac{\partial^2 \gamma_i}{\partial t^2}\frac{\partial^2 v_i}{\partial t^2}$$

$$-M_{11} h^3 \Delta(\kappa_1 + \kappa_2) - C_{11} h^2 \Delta(\varepsilon_1 + \varepsilon_2) + Z = h\rho_{(0)} \frac{\partial^2 w}{\partial t^2} - h^2 \rho_{(1)} \frac{\partial^2}{\partial t^2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) + h^3 \rho_{(2)} \frac{\partial^2}{\partial t^2}\left(\frac{\partial \gamma_1}{\partial x_1} + \frac{\partial \gamma_2}{\partial x_2}\right)$$

$$-M_{11} h^3 \Delta(\kappa_1 + \kappa_2) - C_{11} h^2 \Delta(\varepsilon_1 + \varepsilon_2) + Z = h\rho_{(0)} \frac{\partial^2 w}{\partial t^2} - h^2 \rho_{(1)} \frac{\partial^2}{\partial t^2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) + h^3 \rho_{(2)} \frac{\partial^2}{\partial t^2}\left(\frac{\partial \gamma_1}{\partial x_1} + \frac{\partial \gamma_2}{\partial x_2}\right)$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

We set $C_{11} = 0$

$$C_{11} = \frac{1}{h^2}\int_{z_0}^{z_0+h} \frac{E}{1-\nu^2} x_3 \, \mathrm{d}x_3 = 0, \quad C_{11} = \frac{1}{2h^2}\sum_{k=1}^{n} \frac{E_k (z_k{}^2 - z_{k-1}{}^2)}{1-\nu_k^2} = 0 \tag{9}$$

From the equation (9) we find z_0. The plane $x_3 = 0$ in this coordinate system will be called the neutral plane. In the theory of plates with a gradient of properties, it plays the same role as the middle plane for isotropic plates. The formula (9) for two-layered structure was first obtained in [4].

We rewrite the equations of motion and the elasticity relations of the plate, putting $C_{11} = 0$ in them

$$-M_{11} h^3 \Delta(\kappa_1 + \kappa_2) + Z = h\rho_{(0)} \frac{\partial^2 w}{\partial t^2} - h^2 \rho_{(1)} \frac{\partial^2}{\partial t^2}\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) + h^3 \rho_{(2)} \frac{\partial^2}{\partial t^2}\left(\frac{\partial \gamma_1}{\partial x_1} + \frac{\partial \gamma_2}{\partial x_2}\right) \tag{10}$$

$$G_i = -M_{11} h^3 \kappa_i - M_{12} h^3 \kappa_j - C_{12} h^2 \varepsilon_j, \quad H = 2Mh^3 \tau + Ch^2 \omega \; \varepsilon = O(\eta^1 + \eta^{2-2s})$$

$$A_{11} h \frac{\partial \varepsilon_i}{\partial x_i} + A_{12} h \frac{\partial \varepsilon_j}{\partial x_i} + Ah\frac{\partial \omega}{\partial x_j} + X_i = h\rho_{(0)} \frac{\partial^2 v_i}{\partial t^2} - h^2 \rho_{(1)} \frac{\partial^2 \gamma_i}{\partial t^2} \tag{11}$$

$$T_i = A_{11} h \varepsilon_i + A_{12} h \varepsilon_j + C_{12} h^2 \kappa_j, \quad S = Ah\omega + 2Ch^2 \tau$$

3. Asymptotic analysis of 2D equations

For asymptotic analysis, we turn to dimensionless variables and dimensionless unknown quantities.

As is customary in asymptotic methods, we will perform an asymptotic scale extension with respect to the variable x_i.
We denote the variability of the stress-strain state with respect to the coordinates by s

$$\frac{\partial}{\partial x_i} = \eta^{-s} \frac{1}{R} \frac{\partial}{\partial \xi_i} \tag{12}$$

Suppose that, as in the case of homogeneous plates, the complete problem is conditionally divided into two problems – the problems of quasi transverse vibrations and the quasi tangential vibrations.

3.1. Quasi transverse vibrations
We assume that the vibrations are caused by the uniform normal load Z. Suppose that the deflection is much greater than the tangential displacements $w >> v_i$.
For the unknown quantities we take the following asymptotic representation:

$$\frac{w}{R} = \eta^{\mathbf{0}} w_*, \;\; R\kappa_i = \eta^{-2s}\kappa_{i*}, \;\; R\tau = \eta^{-2s}\tau_*, \;\; \gamma_i = \eta^{-s}\gamma_{i*}, \;\; \frac{hR^4\rho_0}{M_{11}h^2}\frac{\partial^2}{\partial t^2} = \eta^{2-4s}\frac{\partial^2}{\partial t_*^{\,2}} \tag{13}$$

$$\frac{G_i R}{M_{11}h^3} = \eta^{-2s} G_{i*}, \;\; \frac{HR}{M_{11}h^3} = \eta^{-2s} H_*, \;\; \frac{T_i}{A_{11}h} = \eta^{1-2s} T_{i*}, \;\; \frac{SR}{A_{11}h} = \eta^{1-2s} S_*$$

here

$$\kappa_{i*} = \frac{\partial^2 w_*}{\partial \xi_i^2}, \;\; \tau_* = -\frac{\partial \gamma_{1*}}{\partial \xi_2} - \frac{\partial \gamma_{2*}}{\partial \xi_1}, \;\; \gamma_{i*} = -\frac{\partial w_*}{\partial \xi_i}$$

$$G_{i*} = -\kappa_{i*} - \frac{M_{12}}{M_{11}}\kappa_{i*}, \;\; H_* = \frac{2M}{M_{11}}\tau_*, \;\; T_{i*} = \frac{C_{12}}{A_{11}}\kappa_{j*}, \;\; S_* = \frac{2C}{A_{11}}\omega_*$$

In the formulas (13) all quantities with asterisks are dimensionless and of one asymptotic order. We substitute the asymptotic representation of the unknown quantities into equations (10) and perform scale extension (12). As a result we obtain equations with respect to dimensionless unknown quantities, in which the order of each term of the equation is specified. Discarding small terms with the assumed accuracy (1), we obtain the following system of equations:

$$-M_{11}h^3\Delta(\kappa_1^{(b)} + \kappa_2^{(b)}) + Z = h\rho_{(0)}\frac{\partial^2 w^{(b)}}{\partial t^2} - \left[h^2\rho_{(1)}\frac{\partial^2}{\partial t^2}\left(\frac{\partial v_1^{(t)}}{\partial x_1} + \frac{\partial v_2^{(t)}}{\partial x_2}\right)\right] \tag{14}$$

$$G_i^{(b)} = -M_{11}h^3\kappa_i^{(b)} - M_{12}h^3\kappa_j^{(b)}, \;\; H^{(b)} = 2Mh^3\tau^{(b)}$$

$$\kappa_i^{(b)} = \frac{\partial^2 w^{(b)}}{\partial x_i^2}, \;\; \tau^{(b)} = -\frac{\partial \gamma_1^{(b)}}{\partial x_2} - \frac{\partial \gamma_2^{(b)}}{\partial x_1}, \;\; \gamma_i^{(b)} = -\frac{\partial w^{(b)}}{\partial x_i}$$

$$T_i^{(b)} = C_{12}h^2\kappa_j^{(b)}, \;\; S^{(b)} = 2Ch^2\tau^{(b)} \tag{15}$$

The upper indices (b) and (t) indicate the quantities of the quasi transverse and the quasi tangential vibrations problems, respectively.
Equations (14) are equations of quasi transverse vibrations, but the problem can not be considered purely bending, since the tangential forces $T_i^{(b)}$ and $S^{(b)}$ are not equal to zero. In addition, the

equation of motion of a quasi tangential problem contains a term that takes into account the inertia of rotation $h^2\rho_{(1)}\dfrac{\partial^2\gamma_i^{(b)}}{\partial t^2}$.

The sequence of the solution of this problem is as follows: first we solve the bending problem (14), neglecting the term, enclosed in square brackets, then with the help of arithmetic operations we find the forces (15) and the inertia of rotation $h^2\rho_{(1)}\dfrac{\partial^2\gamma_i^{(b)}}{\partial t^2}$.At the second stage we solve the quasi tangential problem with taking into account the quantities found in solving the quasi transverse problem.

3.2. Quasi tangential vibrations

We assume that the vibrations are caused by the uniform tangential load X_i. We consider that the tangential displacements are much greater than the deflections $v_i >> w$.

For the unknown quantities we take the following asymptotic representation:

$$\frac{v_i}{R}=\eta^{\mathbf{0}}v_{i*},\ \ \varepsilon_i=\eta^{-s}\varepsilon_{i*},\ \ \omega=\eta^{-s}\omega_*,\ \ \frac{T_i}{A_{\mathbf{11}}}=\eta^{-s}T_{i*},\ \ \frac{S}{A_{\mathbf{11}}}=\eta^{-s}S_*$$

$$\frac{R^2\rho_{(\mathbf{0})}}{A_{\mathbf{11}}}\frac{\partial^2}{\partial t^2}=\eta^{-2s}\frac{\partial^2}{\partial t_*^{\ 2}},\ \ \frac{G_iR}{M_{\mathbf{11}}h^3}=\eta^{-1-s}G_{i*},\ \ \frac{HR}{M_{\mathbf{11}}h^3}=\eta^{-1-s}H_*$$

Taking into account the asymptotics, we obtain the following equations

$$A_{11}h\frac{\partial\varepsilon_i^{(t)}}{\partial x_i}+A_{12}h\frac{\partial\varepsilon_j^{(t)}}{\partial x_i}+Ah\frac{\partial\omega^{(t)}}{\partial x_j}+X_i=h\rho_{(0)}\frac{\partial^2v_i^{(t)}}{\partial t^2}-\left[h^2\rho_{(1)}\frac{\partial^2\gamma_i}{\partial t^2}\right] \qquad (16)$$

$$\varepsilon_i^{(t)}=\frac{\partial v_i^{(t)}}{\partial x_i},\ \ \omega^{(t)}=\frac{\partial v_{\mathbf{2}}^{(t)}}{\partial x_{\mathbf{1}}}+\frac{\partial v_{\mathbf{1}}^{(t)}}{\partial x_{\mathbf{2}}},\ \ T_i^{(t)}=A_{\mathbf{11}}h\varepsilon_i^{(t)}+A_{\mathbf{12}}h\varepsilon_j^{(t)},\ \ S^{(t)}=Ah\omega^{(t)}$$

$$G_i^{(t)}=-C_{\mathbf{12}}h^2\varepsilon_j^{(t)},\ \ H^{(t)}=Ch^2\omega^{(t)} \qquad (17)$$

As in the case of quasi transverse vibrations, quasi tangential vibrations are related by quasi transverse vibrations through the moments (17) and the inertia of rotation $h^2\rho_{(\mathbf{1})}\dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial v_{\mathbf{1}}^{(t)}}{\partial x_{\mathbf{1}}}+\dfrac{\partial v_{\mathbf{2}}^{(t)}}{\partial x_{\mathbf{2}}}\right)$.

Moments (17) and inertia of rotation are calculated by arithmetic operations after solving the problem for quasi tangential vibrations.

We will solve the complete problem in two stages.

At the first stage we solve the problem (16), (17), discarding in the equations of motion the terms in square brackets.

At the second stage, in the problem for quasi transverse vibrations, we take into account the forces (15) and the inertia of rotation $h^2\rho_{(1)}\dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial v_1^{(t)}}{\partial x_1}+\dfrac{\partial v_2^{(t)}}{\partial x_2}\right)$obtained in the problem for quasi tangential vibrations.

4. Beams

The theory of beams is a special case of the plate theory. The similar results for smart structures were received in [5].
We write the systems of equations for quasi tangential vibrations

$$A_1 h \frac{\partial \varepsilon^{(t)}}{\partial x} + X = h\rho_{(0)} \frac{\partial^2 v^{(t)}}{\partial t^2} - \left[h^2 \rho_{(1)} \frac{\partial^2 \gamma^{(b)}}{\partial t^2} \right], \ \varepsilon^{(t)} = \frac{\partial v^{(t)}}{\partial x}, \ T^{(t)} = Ah\varepsilon^{(t)} \tag{18}$$

and for quasi transverse vibrations

$$-Mh^3 \frac{\partial^2}{\partial x^2} \kappa^{(b)} + Z = h\rho_{(0)} \frac{\partial^2 w^{(b)}}{\partial t^2} - \left[h^2 \rho_{(1)} \frac{\partial^2}{\partial t^2} \frac{\partial v^{(t)}}{\partial x} \right] \tag{19}$$

$$G^{(b)} = -Mh^3 \kappa^{(b)}, \ \kappa^{(b)} = \frac{\partial^2 w^{(b)}}{\partial x^2}, \ \gamma^{(b)} = -\frac{\partial w^{(b)}}{\partial x}$$

The physical constants for a beam with a gradient of properties and for an arbitrary laminated beam are determined by the formulas (20) and (21) respectively

$$A = \frac{1}{h} \int_{z_0}^{z_0+h} E \,\mathrm{d}x_3, \ M = \frac{1}{h^3} \int_{z_0}^{z_0+h} E x_3^2 \,\mathrm{d}x_3, \ \rho_{(0)} = \frac{1}{h} \int_{z_0}^{z_0+h} \rho \,\mathrm{d}x_3, \ \rho_{(1)} = \frac{1}{h^2} \int_{z_0}^{z_0+h} \rho x_3 \,\mathrm{d}x_3 \tag{20}$$

$$A = \frac{1}{h} \sum_{k=1}^{n} E_k h_k, \ M = \frac{1}{3h^3} \sum_{k=1}^{n} E_k ({z_k}^3 - {z_{k-1}}^3), \ \rho_{(0)} = \frac{1}{h} \sum_{k=1}^{n} \rho_k h_k, \ \rho_{(1)} = \frac{1}{2h^2} \sum_{k=1}^{n} \rho_k (z_k - z_{k-1})^2 \tag{21}$$

The position of the neutral axis is found from the equations

$$\sum_{k=1}^{n} E_k ({z_k}^2 - {z_{k-1}}^2) = 0, \ \int_{z_0}^{z_0+h} E x_3 \,\mathrm{d}x_3 = 0 \tag{22}$$

In statics the complete problem for a beam is exactly divided into a plane problem and a bending problem.
In the dynamics, the partition is conditional: quasi transverse (quasi tangential) vibrations generate quasi tangential (quasi transverse) vibrations. The relationship between both kinds of vibrations is realized through the inertia of rotation. Especially dangerous are vibrations with frequencies close to their natural frequencies. In this case, as you approach any natural vibration frequency, all forces, moments, deformations, and displacements increase indefinitely.

5. Numerical examples

5.1. Problem 1

Consider a two-layer beam. The beam with rigidly clamped edges makes vibrations under the action of a uniform tangential load X .The length of the beam is equal to *l*, the thickness is *h*, the layer with the number 1 is three times thinner than the layer with the number 2 ($3h_1 = h_2$). The elasticity modules of the layers and the density of their materials are related by the following relationships

$$E_1 = 3E_2, \ \rho_1 = 2\rho_2$$

We introduce dimensionless variables ξ, ς and dimensionless displacements v_*, w_* and some constants by formulas

$$x_1 = l\xi, \ x_3 = h\varsigma, \ v_* l = v, \ w_* l = w, \ A_* E_2 = A$$

Let us find the position of the neutral axis of the beam

$$\varsigma_0 = \frac{z_0}{h} = -\frac{E^{(1)}h_1^2 + E^{(2)}(h^2 - h_1^2)}{2(E^{(1)}h_1 + E^{(2)}h_2)h} = -0.375$$

$$A = \frac{1}{h}\sum_{k=1}^{2} E_k h_k = 0.75E_2, \quad M = \frac{1}{3h^3}\sum_{k=1}^{2} E_k (z_k{}^3 - z_{k-1}{}^3) = 0.133E_2$$

$$\rho_{(0)} = \frac{\rho_1 h_1 + \rho_2 h_2}{h} = 1.25, \quad \rho_{(1)} = \frac{\rho_1(z_1^2 - z_0^2) + \rho_2(z_2^2 - z_1^2)}{2h^2} = 0.0625, \quad \frac{h}{l} = 0.01, \quad l = 2$$

First we solve problem (18), for which the resolving equation has the form

$$\frac{d^2 v_*}{d\xi^2} + \lambda_1^2 v_* + X_* = 0, \quad X_* = \frac{Xl}{Ah}, \quad \lambda_1^2 = \frac{\rho_0 l^2}{A}, \quad v_* = \frac{\cos\lambda_1\xi}{\cos\lambda_1} - \frac{X_*}{\lambda_1^2}$$

Then calculate the inertia of rotation $h^2\rho_{(1)}\dfrac{\partial^2}{\partial t^2}\dfrac{\partial v^{(t)}}{\partial x}$ and solve the problem (19). The resolving equation of this problem in the dimensionless form is written as follows

$$\frac{d^4 w_*}{d\xi^4} - \lambda_2^4 w_* + \mu^2\lambda_1^2\frac{d v_*}{d\xi} = 0, \quad \lambda_2{}^4 = \frac{Al^2}{Mh^2}\lambda_1{}^2, \quad \mu^2 = \frac{A\rho_{(1)}l}{M\rho_{(0)}h}\lambda_1{}^2$$

The distribution of dimensionless displacement v_* at dimensionless frequency λ=2 along the beam is shown in figure 2 (X_*=1). The calculation shows that under the action of only a tangential load, bending vibrations appeared in the beams.

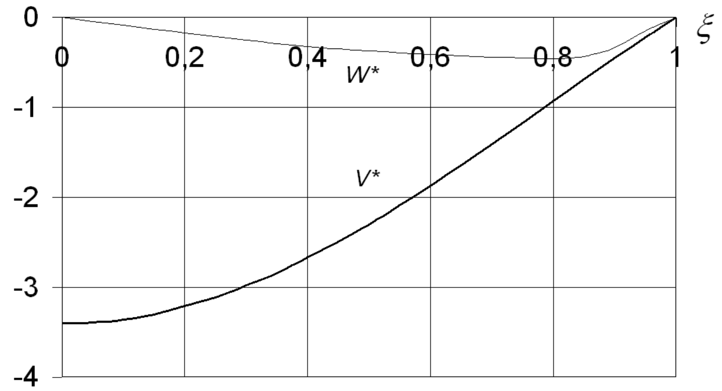

Figure 2. The distribution of dimensionless displacements v_*, w_* along the beam

5.2. Problem 2

We consider axisymmetric forced harmonic vibrations of a circular plate with rigidly clamped edge $r = R$ under the action of a uniform normal load Z. Let the properties of the material of the plate vary in thickness according to a linear law

$$E = (1 + \varsigma_0 - \varsigma)E_0 + (\varsigma - \varsigma_0)E_1, \quad \nu = (1 + \varsigma_0 - \varsigma)\nu_0 + (\varsigma - \varsigma_0)\nu_1, \quad \varsigma h = x_3$$

Here E_0 and ν_0 (E_1 and ν_1) are the values of the modulus of elasticity and the Poisson's ratio on the lower face (the upper face) of the plate.

We take the following values for calculation:

$$E_0 / E_1 = 3, \quad \nu_0 = 0.3, \quad \nu_1 = 0.35, \quad E / E_1 = E_*$$

The first we find the position of neutral plane of the plate by solving equation (9) with respect to ς_0

$$\int_{\varsigma_0}^{\varsigma_0+1} \frac{[(1+\varsigma_0-\varsigma)E_0/E_1+(\varsigma-\varsigma_0)]}{1-[(1+\varsigma_0-\varsigma)\nu_0+(\varsigma-\varsigma_0)\nu_1]^2}\varsigma \mathrm{d}\varsigma = 0$$

Solving the equation for ς_0, we obtain $\varsigma_0 = -0.419$.
We calculate only those constants (5), which are required in solving this problem

$$\frac{M_{11}}{E_1} = 0.1715, \ \frac{M_{12}}{E_1} = 0.7165, \ \frac{C_{12}}{E_1} = 0.0088$$

Here and below, we can take into account that the needed quantities vary as $\exp(-\mathrm{i}\,\omega t)$, ($\omega$ is the circular frequency of vibrations) and write all the equations for the amplitude value only.
The equations for the bending problem in the polar coordinates have the form

$$-M_{11}h^3\Delta(\kappa_r+\kappa_\varphi)+Z+\omega^2 h\rho_{(0)}w=0, \ \ \Delta=\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}$$

$$\gamma_r=\frac{\mathrm{d}w}{\mathrm{d}r}, \ \ \kappa_r=\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}, \ \ \kappa_\varphi=\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}$$

$$G_r=-M_{11}h^3\kappa_r-M_{12}h^3\kappa_\varphi, \ \ G_\varphi=-M_{11}h^3\kappa_\varphi-M_{12}h^3\kappa_r, \ \ T_r=C_{12}\kappa_\varphi, \ \ T_\varphi=C_{12}\kappa_r$$

We now turn to dimensionless unknown quantities and dimensionless coordinates

$$\gamma_{r*}=\frac{\mathrm{d}w_*}{\mathrm{d}\xi}, \ \ \kappa_{r*}=\frac{\mathrm{d}^2 w_*}{\mathrm{d}\xi^2}, \ \ \kappa_{\varphi*}=\frac{1}{\xi}\frac{\mathrm{d}w_*}{\mathrm{d}\xi}, \ \ G_{r*}=-\kappa_{r*}-\frac{M_{12}}{M_{11}}\kappa_{\varphi*}, \ \ G_{\varphi*}=-\kappa_{\varphi*}-\frac{M_{12}}{M_{11}}\kappa_{r*}$$

$$G_{r*}=\frac{G_r}{M_{11}h^3}, \ \ G_{\varphi*}=\frac{G_\varphi}{M_{11}h^3}, \ \ T_{r*}=\frac{C_{12}}{M_{11}}\kappa_{\varphi*}, \ \ T_{\varphi*}=\frac{C_{12}}{M_{11}}\kappa_{r*}$$

$$r=R\xi, \ \ Z_*=\frac{ZR^3}{M_{11}}, \ \ \lambda^2=\frac{\omega^2R^4\rho_{(0)}}{M_{11}h^2}, \ \ w_*=\frac{w}{R}$$

Then we write the resulting equation in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(\frac{\mathrm{d}^2 w_*}{\mathrm{d}\xi^2}+\frac{1}{\xi}\frac{\mathrm{d}w_*}{\mathrm{d}\xi}\right)-Z_*-\lambda^2 w_*=0$$

The general solution of the resulting equation has the form

$$w_*=c_1J_0(\lambda\xi)+c_2I_0(\lambda\xi)-\frac{Z_*}{\lambda^2}$$

where $J_0(\lambda\xi)$ is the Bessel function of the first kind and the zeroth order and $I_0(\lambda\xi)$ is the modified Bessel function of the zeroth order.
We find the integration constants from the conditions at the edge $r=R\,(\xi=1)$

$$w_*=c_1J_0(\lambda)+c_2I_0(\lambda)-\frac{Z_*}{\lambda^2}=0, \ \ \frac{\mathrm{d}w_*}{\mathrm{d}\xi}=\lambda(-c_1J_1(\lambda)+c_2I_1(\lambda))-\frac{Z_*}{\lambda^2}=0$$

$$c_1=\frac{Z_*}{\lambda^2\delta}I_1(\lambda),\ \ c_2=\frac{Z_*}{\lambda^2\delta}J_1(\lambda),\ \ \delta=J_0(\lambda)I_1(\lambda)+J_1(\lambda)I_0(\lambda)$$

From the equation $\delta=0$ we find the first four natural dimensionless frequencies: 3.20, 6.36, 9.44 and 12.58.

Using received formulas, we calculate the bending moment and the tangential force. Imagine the results of the account in the form of graphs. The change in the moment G_{r*} and the force T_{r*} along the radius of the plate is shown in figures 3 and 4 respectively. Calculation is performed for frequencies λ =3 (the dashed line), λ =3.1 (the thin line), and λ =3.15 (the thick line) (Z_* =1).

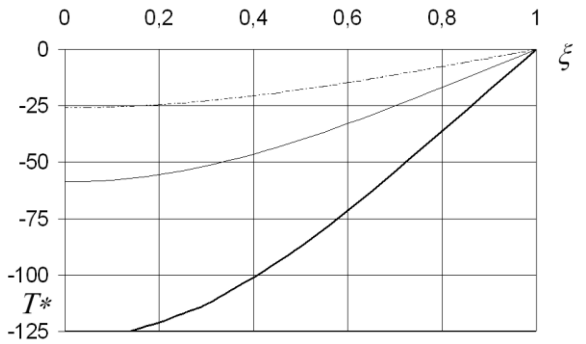

Figure 3. The change in the force T_{r*} along the radius of the plate

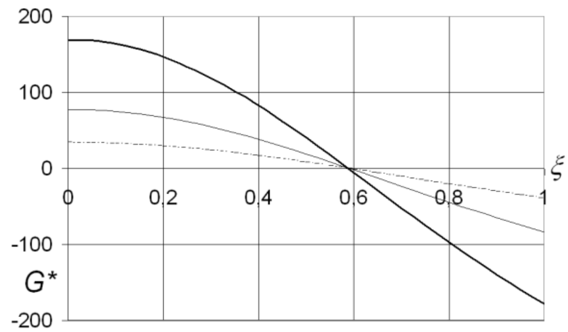

Figure 4. The change in the moment G_{r*} along the radius of the plate

6. Conclusions

We have established the following:

For the thin-walled structures under consideration with asymmetric properties of the material over the thickness, the complete problem in general is not divided into a plane problem and a bending problem. This connectedness of quasi transverse and quasi tangential vibrations leads to an unlimited growth of all unknown quantities (forces, moments, stresses, deformations, displacements) when the frequency of vibrations approaches any natural vibrations frequency. A simple method for calculating the stress-strain state of plates and beams of an asymmetric structure is developed. This method can be used to determine complex material properties.

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