Research of one class of nonlinear differential equations of third order for mathematical modelling the complex structures

Victor Orlov[1] **and Oleg Kovalchuk**[2]

Moscow State University of Civil Engineering, Yaroslavskoye Shosse, 26, Moscow, 129337, Russia

E-mail: [1]OrlovVN@mgsu.ru [2]Kovalchuk@mgsu.ru

Abstract The study of many processes and phenomena in engineering, mechanics, biology, physics, medicine, the vitality of buildings and structures leads to mathematical modeling and the construction of a mathematical model Such models facilitate a rigorous justification of ongoing research and is the only objective option obtain reliable predictions based on analytical methods.

Typically, the mathematical model is one of the aspects of differential equations The basis for more accurate models are nonlinear differential equations In the presence of mobile singular points in the General case it is a class of equations not solvable in quadratures. This circumstance is an obstacle to researchers when constructing mathematical models. This is the importance of the development of the mathematical theory of their decisions.

This paper presents technology and results of solution of some problems of approximate analytical method of solution of nonlinear differential equations with movable singularities:

1 In a complex domain for one class of nonlinear differential equations of third order, not solvable by quadratures, made evidence theorem of existence of solutions in the field of holomorphes.

2 A constructive proof of existence theorems, in contrast to existing classical variants, allows to construct analytical approximate solution of the considered class of nonlinear differential equations, which are used in mathematical modeling of the complex structures.

3 The influence of the perturbation of initial conditions on the obtained analytical approximate solution has been researched.

The obtained results are accompanied by the computational experiments proving their adequacy Solved by the authors of the problem allows you to create mathematical models of complex structures and phenomena. The obtained results allow to carry out the analytical continuation of the approximate solution with a given accuracy In which case, a posteriori error allows obtaining significantly more accurate a priori error.

Keywords: mathematical modelling, nonlinear differential equation, movable singular point

1. Introduction

Analysis of States of various systems includes a stage of mathematical modeling allows predicting the system state under specified impact. A more accurate mathematical model leads to nonlinear differential equations, which require the development of mathematical theory for their solution

As publications attest, differential equations represent mathematical models of various processes and phenomena in the theory of evolutionary equation [1], the theory of elasticity [2], the Kalman Filter theory [3], the nonlinear wave theory [4], nonlinear diffusion [5], mathematical model simulation for building structures [6]-[8] agricultural mechanization [9]. The development of the method of approximate solution of nonlinear differential equations, non-solvable by quadrature in the general case, presented in the papers [10]-[14], enables researchers to simulate more accurate mathematical models of investigated processes and phenomena based on nonlinear differential equations.

2. Study materials and research design

In [15], taking into account the necessary condition for the presence of a movable singular point [16], their existence is proved, and an analytic approximate solution is constructed in the neighborhood of the movable singular point.

In the present paper, for the considered class of nonlinear differential equations in a complex domain the existential theorem in holomorphic region has been proved, analytical approximate solution has been designed as well as the influence of perturbation of initial conditions on the latter has been researched. This problem arises when we apply analytic continuation of the solution. It should be pointed out that the existing classical Cauchy's theorem doesn't provide means for obtaining the presented results.

3. The findings of the investigation

Consider the nonlinear differential equation

$$y''' = a_0(z)y^2 + a_1(z)y + a_2(z).$$

The paper [14] suggests that the equation is reduced to a normal form by making the change of variable. Let us consider Cauchy problem

$$y''' = y^2 + r(z) \tag{1}$$

$$y(z_0) = y_0, \quad y'(z_0) = y_1, \quad y''(z_0) = y_2 \tag{2}$$

3.1. Theorem 1.

Suppose we are given hypothesis:

1) $r(z) \in C^1$ in the region $|z - z_0| < \rho_1$, $\rho_1 = \text{const}$;

2) $\exists M_n : \left|\frac{r^{(n)}(z_0)}{n!}\right| \le M_n$, $n = 0, 1, 2, \ldots$

Then y(z) is holomorphic function

$$y(z) = \sum_0^\infty C_n (z - z_0)^n \tag{3}$$

in the region $|z - z_0| < \rho_2$, where

$$\rho_2 = \min\left\{\rho_1, \frac{1}{\sqrt[3]{M+1}}\right\}, \quad M = \max\left\{|y_0|, |y_1|, |y_2|, \sup_n \left|\frac{r^{(n)}(z_0)}{n!}\right|\right\}.$$

3.1.1. Theorem proving Under hypothesis of the theorem we have

$$r(z) = \sum_0^\infty A_n (z - z_0)^n \tag{4}$$

Let's substitute (3) and (4) into (1):

$$\sum_{3}^{\infty} n(n-1)(n-2)C_n(z-z_0)^{n-3} = \sum_{0}^{\infty} C_n^*(z-z_0)^n + \sum_{0}^{\infty} A_n(z-z_0)^n ,$$

where $C_n^* = \sum_{i=0}^{n} C_{n-i}C_i$. The latter will be the identity provided

$$n(n-1)(n-2)C_n = C_{n-3}^* + A_{n-3}, \qquad n = 3, 4, \ldots \tag{5}$$

Recurrence relationship (5) allows obtaining the expressions for the coefficients C_n on a personal computer. The reached expressions suggest the hypothesis for the estimated coefficient C_n:

$$|C_{3n}| \le \frac{(M+1)^{n+1}}{3n(3n-1)(3n-2)}, \quad |C_{3n+1}| \le \frac{(M+1)^{n+1}}{3n(3n+1)(3n-1)}, \quad |C_{3n+2}| \le \frac{(M+1)^{n+1}}{3n(3n+1)(3n+2)} \tag{6}$$

We now show validity of the estimated coefficients C_{3n+3} . From the relationship (5) follows

$$(3n+1)(3n+2)(3n+3)C_{3n+3} = C_{3n}^* + A_{3n} .$$

Then, with the due account of the hypothesis (6) we receive

$$|C_{3n+3}| \le \frac{1}{(3n+1)(3n+2)(3n+3)} \left(\sum_{i=0}^{n} \frac{(M+1)^{n+1-i}}{(3n-3i)(3n-3i-1)(3n-3i-2)} \frac{(M+1)^{i+1}}{(3i(3i-1)(3i-2))^*} + M \right) \le$$

$$\le \frac{(M+1)^{n+2}}{(3n+1)(3n+2)(3n+3)} ,$$

where

$$(3i(3i-1)(3i-2))^* = \begin{cases} 1, \ i = 0, \\ 3i(3i-1)(3i-2), \ i = 1, 2, \ldots . \end{cases}$$

By the same procedure we ascertain in the estimated coefficients C_{3n+1} and C_{3n+2}
Let us consider series

$$\sum_{n=1}^{\infty} \frac{(M+1)^{n+1}|z-z_0|^{3n}}{3n(3n-1)(3n-2)} ,$$

for which we receive the region on sure ground of a sufficient condition

$$|z-z_0| < \frac{1}{\sqrt[3]{M+1}} .$$

Thus, due to specific characteristics of the estimated coefficients (6) C_n in (3) for the series in the formula (3), we receive the region

$$|z-z_0| < \rho_2 ,$$

where $\rho_2 = \min\left\{ \rho_1, \frac{1}{\sqrt[3]{M+1}} \right\}$.

The structure of the approximate solution is determined by *Theorem 1*

$$y_N(z) = \sum_{0}^{N} C_n (z-z_0)^n \tag{7}$$

3.2. Theorem 2.

Under the preceding hypothesis 1 and 2 of the theorem 1**,** for approximate solution (7) the problems (1) - (2) in the region $|z-z_0| < \rho_2$ in the case of $N+1 = 3n$ the estimate is valid

$$\Delta y_N(z) \le \frac{(M+1)^{\frac{N+4}{3}} |z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)} \left(\frac{1}{N(N-1)} + \frac{|z-z_0|}{N(N+2)} + \frac{|z-z_0|^2}{(N+2)(N+3)} \right),$$

for $N+1 = 3n+1$ the valid estimate is as follows

$$\Delta y_N(z) \le \frac{(M+1)^{\frac{N+3}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{N(N-1)}+\frac{|z-z_0|}{N(N+2)}+\frac{(M+1)|z-z_0|^2}{(N+2)(N+3)}\right)$$

and for the variant $N+1=3n+2$ the estimate is as follows:

$$\Delta y_N(z) \le \frac{(M+1)^{\frac{N+2}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{N(N-1)}+\frac{(M+1)|z-z_0|}{N(N+2)}+\frac{(M+1)|z-z_0|^2}{(N+2)(N+3)}\right),$$

where ρ_2 and M – from the *Theorem 1* .

3.2.1. Theorem proving. By definition

$$\Delta y_N(z) = |y-y_N(z)| = \left|\sum_{N+1}^{\infty} C_n (z-z_0)^n\right|.$$

Thus, due to specific characteristics of the estimated coefficients C_n in (6) in the case $N+1=3n$ we'll have

$$\Delta y_N(z) = \left|\sum_{k=0}^{\infty} C_{3(n+k)}(z-z_0)^{3(n+k)} + \sum_{k=0}^{\infty} C_{3(n+k)+1}(z-z_0)^{3(n+k)+1} + \sum_{k=0}^{\infty} C_{3(n+k)+2}(z-z_0)^{3(n+k)+2}\right| \le$$

$$\le \sum_{k=0}^{\infty}\frac{(M+1)^{n+k+1}|z-z_0|^{3(n+k)}}{3(n+k)(3(n+k)-1)(3(n+k)-2)} + \sum_{k=0}^{\infty}\frac{(M+1)^{n+k+1}|z-z_0|^{3(n+k)+1}}{(3(n+k)+1)3(n+k)(3(n+k)-1)} +$$

$$+\sum_{k=0}^{\infty}\frac{(M+1)^{n+k+1}|z-z_0|^{3(n+k)+2}}{(3(n+k)+2)(3(n+k)+1)3(n+k)} \le$$

$$\le \frac{(M+1)^{\frac{N+4}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{N(N-1)}+\frac{|z-z_0|}{N(N+2)}+\frac{|z-z_0|^2}{(N+2)(N+3)}\right).$$

Analogously, in the case $N+1=3n+1$ we have the estimate

$$\Delta y_N(z) \le \frac{(M+1)^{\frac{N+3}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{N(N-1)}+\frac{|z-z_0|}{N(N+2)}+\frac{(M+1)|z-z_0|^2}{(N+2)(N+3)}\right),$$

and for the variant $N+1=3n+2$ the valid estimate is:

$$\Delta y_N(z) \le \frac{(M+1)^{\frac{N+2}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{N(N-1)}+\frac{(M+1)|z-z_0|}{N(N+2)}+\frac{(M+1)|z-z_0|^2}{(N+2)(N+3)}\right).$$

The presented estimates of approximate solution are valid in the region $|z-z_0|<\rho_2$, where $\rho_2 = \min\left\{\rho_1, \frac{1}{\sqrt[3]{M+1}}\right\}$.

3.2.2. The example 1. Let us consider Cauchy problem.

$$y''' = y^2(z)+z, \quad y(1+i)=0{,}5+0{,}3i, \quad y'(1+i)=0{,}5+0{,}5i,$$

$$y''(1+i)=1+i, \quad z_1 = 1{,}4+1{,}3i, \quad \rho_2 = 0{,}5848035.$$

Numerical calculations are set out in the **Table 1**.

Table 1. Numerical calculations of approximate solution in the case of accurate initial conditions

z_1	$y(z_0)$	$y'(z_0)$	$y''(z_0)$	$y_8(z)$	Δ_1	Δ_2
1.4+1.3	0.5+0.3i	0.5+0.5i	1+ i	$0.342557+0.971656i$	0.000237	$6\cdot10^{-6}$

Here $y_8(z)$ – approximate solution of Cauchy problem (1) - (2); Δ_1 – prior error estimate according to the *Theorem 2*; Δ_2 – posterior error estimate for which on the basis of the results of the *Theorem 2* the value $N=14$ is necessary. Summands from 9 to 14 in the structure of approximate solution do not exceed the specified accuracy $\varepsilon=6\cdot10^{-6}$. It follows that the value $y_8(z)$ is obtained with the accuracy $\varepsilon=6\cdot10^{-6}$.

While implementing analytical continuation of the solution of the problems (1) - (2), we arrive at the task of investigation of influence of initial conditions perturbation of the data of Cauchy problem

$$y'''=y^2+r(z) \tag{8}$$

$$y(z_0)=\tilde{y}_0 \quad y'(z_0)=\tilde{y}_1 \quad y''(z_0)=\tilde{y}_2 \tag{9}$$

on the structure of analytical approximate solution (7) which takes the form of

$$\tilde{y}_N(z)=\sum_0^N \tilde{C}_n(z-z_0)^n \tag{10}$$

where $\tilde{C}_n$ – perturbed values of the coefficients The following theorem allows error estimating analytical approximate solution (10)

3.3. Theorem 3.

Under 1) the preceding hypothesises 1 and 2 theorem 1; 2) $\Delta M\le 1$. Then for analytical approximate solution (10) for the problems (8) - (9) in the region $|z-z_0|<\rho_3$, the valid estimate is as follows:

$$\Delta\tilde{y}_N(z)\le\Delta_0+\Delta_1,$$

where

$$\Delta_0\le\frac{\Delta M(M+\Delta M+1)}{1-(M+\Delta M+1)|z-z_0|^3}(1+|z-z_0|+|z-z_0|^2),$$

Δ_1 in the case $N+1=3n$ has an expression

$$\Delta_1\le\frac{(M+1)^{\frac{N+4}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{(N+1)N}+\frac{|z-z_0|}{N(N+2)}+\frac{|z-z_0|^2}{(N+2)(N+3)}\right),$$

in the case $N+1=3n+1$ we get

$$\Delta_1\le\frac{(M+1)^{\frac{N+3}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{(N+1)N}+\frac{|z-z_0|}{N(N+2)}+\frac{(M+1)|z-z_0|^2}{(N+2)(N+3)}\right)$$

and for the variant $N+1=3n+2$ the valid estimate is

$$\Delta_1\le\frac{(M+1)^{\frac{N+2}{3}}|z-z_0|^{N+1}}{(1-(M+1)|z-z_0|^3)(N+1)}\left(\frac{1}{(N+1)N}+\frac{(M+1)|z-z_0|}{N(N+2)}+\frac{(M+1)|z-z_0|^2}{(N+2)(N+3)}\right).$$

At that

$$M=\max\left\{|\tilde{y}_0|,|\tilde{y}_1|,|\tilde{y}_2|,\sup_n\left|\frac{r^{(n)}(z_0)}{n!}\right|\right\},$$

$$\Delta M=\max\{\Delta\tilde{y}_0,\Delta\tilde{y}_1,\Delta\tilde{y}_2\},\qquad \rho_3=\min\left\{\rho_1,\frac{1}{\sqrt[3]{M+\Delta M+1}}\right\}.$$

3.3.1. Theorem proving. By definition

$$\Delta\tilde{y}_N(z)=|y(z)-\tilde{y}_N(z)|\le|y(z)-\tilde{y}(z)|+|\tilde{y}(z)-\tilde{y}_N(z)|=$$

$$=\left|\sum_0^\infty C_n(z-z_0)^n-\sum_0^\infty\tilde{C}_n(z-z_0)^n\right|+\left|\sum_0^\infty\tilde{C}_n(z-z_0)^n-\sum_0^N\tilde{C}_n(z-z_0)^n\right|\le$$

$$\le\sum_0^\infty\Delta\tilde{C}_n|z-z_0|^n+\sum_{N+1}^\infty|\tilde{C}_n||z-z_0|^n=\Delta_0+\Delta_1,$$

where $\Delta\tilde{C}_n=|C_n-\tilde{C}_n|$. Taking into account the estimates of the coefficients C_n in (6), we form a hypothesis to estimate $\Delta\tilde{C}_n$:

$$\left.\begin{array}{c}\Delta\tilde{C}_{3n}\le\dfrac{\Delta M(M+\Delta M+1)^{n+1}}{3n(3n-1)(3n-2)},\qquad \Delta\tilde{C}_{3n+1}\le\dfrac{\Delta M(M+\Delta M+1)^{n+1}}{(3n-1)3n(3n+1)},\\ \Delta\tilde{C}_{3n+2}\le\dfrac{\Delta M(M+\Delta M+1)^{n+1}}{3n(3n+1)(3n+2)}.\end{array}\right\}\tag{11}$$

We now show the truth of the hypothesis for an estimate $\Delta\tilde{C}_{3n}$. With due account for recurrence relationship (5) we have

$$\Delta\tilde{C}_{3n+3}=|C_{3n+3}-\tilde{C}_{3n+3}|=\frac{1}{(3n+1)(3n+2)(3n+3)}\left|C_{3n}^*+A_{3n}-\tilde{C}_{3n}^*-A_{3n}\right|=$$

$$=\frac{1}{(3n+1)(3n+2)(3n+3)}\left|\sum_{i=0}^\infty(\tilde{C}_{3n-3i}+\Delta\tilde{C}_{3n-3i})(\tilde{C}_{3i}+\Delta\tilde{C}_{3i})-\sum_{i=0}^\infty\tilde{C}_{3n-3i}\tilde{C}_{3i}\right|\le$$

$$\le\frac{1}{(3n+1)(3n+2)(3n+3)}\sum_{i=0}^\infty\left(\frac{(M+1)^{n-i+1}\Delta M(M+\Delta M+1)^{i+1}}{(3n-3i)(3n-3i-1)(3n-3i-2)(3i(3i-1)(3i-2))^*}+\right.$$

$$+\frac{\Delta M(M+\Delta M+1)^{n-i+1}(M+1)^{i+1}}{(3n-3i)(3n-3i-1)(3n-3i-2)(3i(3i-1)(3i-2))^*}+$$

$$\left.+\frac{\Delta M(M+\Delta M+1)^{n-i+1}}{(3n-3i)(3n-3i-1)(3n-3i-2)}\frac{\Delta M(M+\Delta M+1)^{i+1}}{(3i(3i-1)(3i-2))^*}\right)\le\frac{\Delta M(M+\Delta M+1)^{n+2}}{(3n+1)(3n+2)(3n+3)},$$

where

$$(3i(3i-1)(3i-2))^*=\begin{cases}1, & i=0,\\ 3i(3i-1)(3i-2), & i=1,2,\ldots.\end{cases}$$

In the same manner we verify that the estimates for $\Delta\tilde{C}_{3n+1}$ and $\Delta\tilde{C}_{3n+2}$ are accurate. Then

$$\Delta_0=\sum_0^\infty\Delta\tilde{C}_n|z-z_0|^n=\sum_0^\infty\Delta\tilde{C}_{3n}|z-z_0|^{3n}+\sum_0^\infty\Delta\tilde{C}_{3n+1}|z-z_0|^{3n+1}+\sum_0^\infty\Delta\tilde{C}_{3n+2}|z-z_0|^{3n+2}.$$

Taking into account the estimates (11), we finally receive for Δ_0

$$\Delta_0\le\frac{\Delta M(M+\Delta M+1)}{1-(M+\Delta M+1)|z-z_0|^3}(1+|z-z_0|+|z-z_0|^2).$$

The estimate for Δ_1 follows from the *Theorem 2.*

The proven estimates of approximate solution are valid in the region of $|z-z_0|<\rho_3$, where $\rho_3=\min\left\{\rho_1,\frac{1}{\sqrt[3]{M+\Delta M+1}}\right\}$.

Analysing the structure of the expressions for Δ_0 and Δ_1, the following conclusions could be made: the expression for Δ_0 depends on the value of perturbation of initial conditions while the expression Δ_1 is linked to the structure of analytical approximate solution. Combining these parameters in a certain manner, it's possible to receive the value of approximate solution with the specified accuracy.

3.3.2. The example 2. Let us consider Cauchy problem with perturbed initial conditions.

$$y'''=y^2(z)+z, \quad \tilde{y}(1.4+1.3i)=0.342557+0.971656i,$$
$$\tilde{y}'(1.4+1.3i)=0.553761+2.089490i,$$
$$\tilde{y}''(1.4+1.3i)=1.914639+2.977412i, \quad z_2=1.8+1.7i, \quad \rho_3=0.603928$$

The calculations are set out in the **Table 2**.

Table 2. The calculations of approximate solution in the case of perturbed initial conditions

z_2	$\tilde{y}(z_1)$	$\tilde{y}'(z_1)$	$\tilde{y}''(z_1)$	$\tilde{y}_8(z)$	Δ_1	Δ_2
1.8+1.7	0.342557+ +0.97165*i*	0.553761+ +2.089490	1.914639 +2.97741	-1.269458+ +2.605812*i*	0.00855	$7\cdot 10^{-5}$

Here Δ_1 – prior error estimate on the basis of the *Theorem 3* results; Δ_2 – posterior error estimate with the value $N=38$ as required in the structure of approximate solution according to the *Theorem 3.* The summands of approximate solution from 9 to 38 do not exceed the specified accuracy. In this way, it is fair to say that approximate solution $\tilde{y}_8(z)$ has an operational margin not exceeding the value $\varepsilon=7\cdot10^{-5}$.

4. Conclusion

Solved by the authors of the problem allows you to create mathematical models of complex structures and phenomena. The obtained results allow to carry out the analytical continuation of the approximate solution with a given accuracy. In which case, a posteriori error allows obtaining significantly more accurate a priori error.

References

[1] Airault H 1979 Rational Solutions of Painleve Equations Studies in applied mathematics vol 61 No 1 pp 31–53
[2] Hill J M 1977 Radial deflections of thin precompressed cylindrical rubber bush mountings Internat Solids Structures vol 13 pp 93–104
[3] Kalman R 1960 Contribution to the theory of optimal control Boletin de la Sociedad Matematica Mehanica Segunde serie vol 5 No 1 pp 102–119
[4] Hill J M 1982 Abel's Differential Equation Math Scientist vol 7 No 2 pp 115–125
[5] Ockendon J R 1978 Numerical and analytical solutions of moving boundary problems Proc. Symp. Moving Boundary Problems Ed. D.G. Wilson, A.D. Solomon and P.T. Boggs. (New York) pp 129–145

IOP Conf. Series: Materials Science and Engineering **365** (2018) 042045 doi:10.1088/1757-899X/365/4/042045

[6] Kovalchuk O A 2016 Simulation of the State of the Rod Elements of the Building Construction Procedia Engineering vol 153 No 2 pp 304–309
[7] Kovalchuk O A 2016 Industrial and Civil Engineering No 9 (Moscow) pp 70–73
[8] Kovalchuk O A 2014 Industrial and Civil Engineering No 11 (Moscow) pp 53–54
[9] Gorin V A, Konakov A P and Popov N S 1981 Mechanization and Electrification of Agriculture No 1 (Moscow) pp 24–26
[10] Orlov V N 2013 The Method of Approximate Solution of First and Second Orders of Painleve and Abel Differential Equations (Moscow: Moscow state pedagogical University Press) p 173
[11] Orlov V N 2009 Bulletin of Bauman Moscow state technical University Estestvennye nauki (Moscow) No 4(35) pp 23–32
[12] Orlov V N 2009 Bulletin of Voronezh state technical University (In Russia: Voronezh) vol 5 No 10 pp 192–195
[13] Redkozubov S A and Orlov V N 2009 Izvestiya instituta inzhenernoj fiziki (Moscow) No 4(14) pp 12–14
[14] Orlov V N 2010 Bulletin of the Chuvash state University named after Yakovlev Mekhanika predel'nogo sostoyaniya (In Russian: Cheboksary) No 2(8), pp 399–405
[15] Pchelova A Z and Kolle K V 2015 Conf. Mechanics of Limiting State and related topics dedicated to 85th Anniversary of prof Ivlev D D (In Russian: Cheboksary) Vol 2 No 2 pp 221–226
[16] Orlov V N 2013 Conf. Fundamental and Applied Issues of Mechanics of Deformable Solids, Mathematical Modelling and Information Technologies (In Russian: Cheboksary) Vol 2 No 2 pp 30–35