IOP Conf. Series: Materials Science and Engineering **360** (2018) 012041 doi:10.1088/1757-899X/360/1/012041

A comparative study on the postbuckling behaviour of circular plates

Anju V Nair[1], Abdul Rahman Mohd Kasim[1*], Mohd Zuki Salleh[1]

[1]Faculty of Industrial Sciences and Technology, Universiti Malaysia Pahang

Lebuhraya Tun Razak, 26300, Kuantan, Malaysia

Corresponding author: rahmanmohd@ump.edu.my

Abstract. Materials/metals, polymers, reinforced plastics, composites, functionally graded materials, ceramics and nano materials find significant place in many engineering and scientific applications. The material structural members such as plates and beams are subjected to structural deformation owing to several external and internal factors such as temperature, hysteresis and so on. Therefore, to analyze the structural deformation of these material structures, its nonlinear behavior modeling is relevant. The rapid progress in high speed computing devices resolve research problems involving material non-linearity behavior such as post-buckling behavior in varied structures. The post-buckling behavior of commonly used structural elements especially plates under thermal and mechanical loads are of great practically important research topic. Therefore, a comparative study on the thermal post buckling behavior of plates by various numerical methods such as finite element formulation, shear deformation theory, Rayleigh – Ritz method, iteration methods and so on will be helpful to obtain a cutting edge on numerical solutions and their applications in varied engineering applications. This paper reports a novel mathematical formulation involving substitution method to evaluate and compare the thermal post buckling load carrying capacity of circular plates with minimal errors.

1. Introduction

Study of deformation of structural members was based on high margins of safety. The effects due to nonlinearities involved in either strain – displacement relations termed as geometric nonlinearity, stress-strain relations termed as material nonlinearity is gaining research importance in field of structural mechanics. The nonlinearity in the behavior of the material is termed as material nonlinearity or physical nonlinearity. The analysis of structural members with physical nonlinearity is gaining research importance because of introduction of new materials such as metals, polymers, reinforced plastics, composites, functionally graded materials, ceramics and nano materials in aerospace and other industries. Application of these materials in the structural members under severe thermo-mechanical loads is of prime concern for researchers.

Beams, rectangular plates and circular plates (thin and thick) are commonly used structural elements in idealizing large aerospace structures. During their service conditions, these members/elements are subjected to severe thermo mechanical loads owing to aerodynamic/solar heating. Therefore, the effect of nonlinearities (geometric and material) in structural members when subjected to the loading conditions as close to reality as possible has to be analysed mathematically. Numerical formulations is proved to be a simple and powerful mathematical technique for designers, practicing engineers as well as for researchers who wish to have accurate engineering results with less mathematical intervention and computational effort. Most of the earlier studies have been using the classical methods/energy methods/ variational methods and the versatile finite element method. However, the solution procedure in all the above mentioned studies involves solving of higher mathematical terms and solving higher order differential equations or involve in cumbersome calculations.

In the present study, the prime focus is on the post-buckling analysis of circular plates considering a wide range of loading conditions. Similar to other plates, the post-buckling of circular plate also occurs owing to in-plane loading. The precise knowledge of critical buckling loads, mode shapes and the post buckling behaviour is important for a reliable circular plate design. The simply supported and clamped boundary conditions are used in most of the post-buckling studies of circular plates. Different numerical methods such as finite element approach, iteration methods, and shear deformation theory has been developed owing to the variation in the thickness of circular plates.

Friedrichs and stoker [1] explained the post buckling behaviour of a simply supported circular plate at large deflection, by solving the coupled nonlinear von Karman equations, using power series and perturbation method. The method is extended to clamped circular plates by Bodner [2]. The thermal post buckling behaviour of isotropic circular plates with simply supported and clamped boundary conditions using finite element analysis has been presented in [3]. The numerical results are compared with the Rayleigh Ritz solutions to check the accuracy of the results. Table 1 compares the linear bucking load value of circular plates under simply supported and clamped boundary conditions through different numerical analysis techniques. This linear buckling load is used to evaluate the post-buckling load of circular plates under mentioned boundary conditions and further explained in the mathematical formulation part of the manuscript. This manuscript reports a novel numerical technique to determine the thermal post buckling load of circular plates and to best of our knowledge the error percentage for the calculated value is minimal to other reported post-bucking values for circular plates.

Table 1: Tabulation of numerical values of linear buckling load with simply supported and clamped boundary conditions.

Ref.	**Contents**	**Linear buckling load for simply supported circular plates**	**Linear buckling load for clamped circular plates**
[3]	The thermal post buckling behavior of isotropic circular plates has been studied using finite element formulation and compared with Rayleigh Ritz method	4.1978	14.6896
[4]	The post buckling behavior of linearly tapered circular plates for different taper parameters having orthotropic properties has been investigated using finite element formulation	4.1978	14.6826
[5]	A simple finite element formulation for the post buckling analysis of circular plates has been presented. The approximate solutions are evaluated by using (i) iterative method (ii) step by step method and (iii) linear vector method.	4.198	14.69
[6]	The post buckling behaviour of elastic circular plates are reinvestigated using a modified finite element analysis.	4.1978	14.6894

[7]	A finite element method is used to analyse the post buckling behaviour of circular plates with elastically restrained edges under thermal loads.	4.1978	14.6826
[8]	A simple formulation to predict the thermal post buckling load of circular plates by evaluating the radial edge tensile load has been presented.	4.1978	14.6896
[9]	Based on radial edge tensile load a novel formulation to study the post buckling behaviour of circular plates elastically restrained edges against rotation has been presented.	4.1977	14.6986
[10]	The large amplitude vibrations of circular plates has been investigated by using a modified Berger's approximation. The governing equations are considered based on Von Karman nonlinearities.	-	-
[11]	The post buckling behaviour of orthotropic circular plates using a simple finite element has been presented. The numerical results are described in the form of radial load ratios with their corresponding empirical formulae.	4.1979	14.6896

2. Mathematical formulation

The objective of this section is to develop a mathematical formulation to evaluate the thermal post buckling load carrying capacity of the circular plates. By considering a circular plate of radius 'a' and of uniform thickness 't' under a radial uniform compressive load 'N_r' per unit length at the boundary, the von Karman type nonlinear strain – displacement relations of the circular plate for large lateral axisymmetric displacements are

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (1); \qquad \varepsilon_\theta = \frac{u}{r} \quad (2); \qquad \chi_r = -\frac{d^2w}{dr^2} \quad (3); \qquad \chi_\theta = -\frac{1}{r}\left(\frac{dw}{dr}\right) \quad (4)$$

where r and θ are the radial and circumferential coordinates, $\varepsilon_r, \varepsilon_\theta$ are the strains and χ_r, χ_θ are the curvatures.

The strain energy U of the plate with orthotropic material properties, is given by

$$U = \frac{1}{2}\int_0^{2\pi}\int_{r_1}^{r_2}\left[C_1\varepsilon_r^{\,2} + C_2\,\varepsilon_\theta^{\,2} + C_{12}\,\varepsilon_r\varepsilon_\theta + D_1\,\chi_r^{\,2} + D_2\,\chi_\theta^{\,2} + D_{12}\,\chi_r\chi_\theta\right] r\,dr\,d\theta \quad (5)$$

where $C_1 = \frac{E_r h}{1-\nu^2}$, $C_2 = \frac{E_\theta h}{1-\nu^2}$, $C_{12} = \nu_r C_2 = \nu_\theta C_1$, $D_1 = \frac{E_r h^3}{12\left(1-\nu^2\right)}$, $D_2 = \frac{E_\theta h^3}{12\left(1-\nu^2\right)}$, and $D_{12} = \nu_r D_2 = \nu_\theta D_1$

Using appropriate substitutions, eqn. (5) can be written as

$$U = \frac{1}{2}\int_0^1\left(\left(\frac{d^2w}{dr^2}\right)^2 + \beta\frac{1}{r^2}\left(\frac{dw}{dr}\right)^2 + 2\nu\frac{\beta}{r^2}\left(\frac{d^2w}{dr^2}\right)\left(\frac{dw}{dr}\right)\right)dr \quad (6)$$

for the orthotropic parameter $\beta = {E_\theta}/{E_r}$. If $E_\theta = E_r$ (β = 1), then the plates are isotropic and otherwise the plates are orthotropic ($E_\theta \neq E_r$).

The work done, W by the external load N_r per unit length at the boundary of the element is given by

$$W = \frac{1}{2}\int_0^{2\pi}\int_{r_1}^{r_2}\overline{N}_r\left(\frac{dw}{dr}\right)^2 r\,dr\,d\theta \quad (7)$$

By using substitution, equation (7) can be reduced to

$$W = \frac{\lambda\beta}{2}\int_0^1 \frac{d^2w}{dr^2}\,dr \tag{8}$$

The total potential energy, is the function of in – plane displacement fields and lateral displacement fields of the plate can be stated as $\Pi = U - W$ (9)

By assuming suitable admissible function for the lateral displacement '*w*', which satisfies the geometric boundary conditions, the uniform radial edge tensile load N_{r_T} is obtained. The value of the Poisson's ratio ν is taken as 0.3 and both simply supported and clamped boundary conditions are considered in this study.

The following admissible function F taken from [12] for the lateral displacement '*w*' is used to study the thermal post buckling behavior of circular plates.

$$F = b_0\left[1+\alpha_1\left(\frac{r}{a}\right)^2+\alpha_2\left(\frac{r}{a}\right)^4\right]+b_1\left[\alpha_3\left(\frac{r}{a}\right)^2+\alpha_4\left(\frac{r}{a}\right)^4+\left(\frac{r}{a}\right)^6\right]+b_2\left[\alpha_5\left(\frac{r}{a}\right)^4+\alpha_6\left(\frac{r}{a}\right)^6+\left(\frac{r}{a}\right)^8\right] \tag{10}$$

The values of α_1, α_2, α_3, α_4, α_5 and α_6, are given in terms of ν and β represent the simply supported and clamped boundary conditions for the function F. By solving equation (9), one can get the eigen values and eigen vectors with appropriate boundary conditions and can evaluate the linear buckling load parameters. The total equivalent compressive uniform radial edge load carrying capacity of the circular plate $N_{r_{NL}}$ can be mathematically represented in the non-dimensional form, as

$$\overline{N}_{r_{NL}} = \overline{N}_{r_{Cr}} + \overline{N}_{r_T} \tag{11}$$

where each term in equation (11) is non – dimensionalised as $\overline{N}_r = \dfrac{N_r a^2}{D}$ (12)

and D is the plate flexural rigidity.

From equations (1) and (2), the radial edge tensile load per unit length at any point r on the radius is evaluated as

$$N_r = \frac{\beta t}{1-\nu^2}\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu\frac{u}{r}\right] \tag{13}$$

By using the approximation proposed by Berger [13] which states that the second invariant of the strains are neglected or $\varepsilon_r << \varepsilon_\theta$, N_r can be written as

$$N_r = \frac{\beta t}{1-\nu^2}\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]. \tag{14}$$

The uniform radial tensile load developed in the circular plate due to large deflections, can be obtained in the non – dimensional form as

$$N_{r_T} = \frac{12}{\beta\left(1-\nu^2\right)}\int_0^1\left(\frac{dw}{dr}\right)^2 dr \tag{15}$$

The post buckling load carrying capacity (γ), is obtained from equation (11) using the values of radial edge tensile load parameters evaluated from the assumed admissible functions for the lateral displacement w and the linear buckling load parameters as

$$\gamma = \frac{N_{r_{NL}}}{N_{r_{cr}}} = 1 + c\left(\frac{b_0}{t}\right)^2 \tag{16}$$

The values of γ are evaluated for different values of b_0/t for $\beta = 1$ which represents isotropic circular plates.

3. Numerical results and discussions

The radial edge tensile load and the thermal post buckling load carrying capacity is evaluated by assuming suitable admissible function for the lateral displacement which satisfies both simply supported and clamped boundary conditions and the numerical results obtained with varying values of b_0/t for β = 1.0. Tables 2 and 3 represent the post buckling load carrying capacity of isotropic circular plates with both simply supported and clamped boundary conditions. The numerical results from the known literatures are compared including the results of the present formulation. It can be shown that the values of the post buckling load carrying capacity of the isotropic plates are in good agreement. The percentage errors from the reference values are also evaluated and it can be shown that the highest error percentage is 1.41% for simply supported and 1.66% for clamped boundary conditions. The error is occurred in the values of γ due to the edge boundary conditions and depends on the admissible function for the lateral displacement.

Table 2: Representing the values of post buckling load carrying capacity of simply supported isotropic circular plates.

b_0/t	**Post buckling load carrying capacity (γ)**						
	PRESENT	REF [3]	REF [14]	REF [15]	Percentage error from REF [3]	Percentage error from REF [14]	Percentage error from REF [15]
0.0	1.0000	1.0000	1.0000	1.0000	0	0	0
0.2	1.0820	1.0730	1.0706	1.0730	0.83	1.06	0.83
0.4	1.3042	1.2931	1.3043	1.2930	0.85	0.01	0.86
0.6	1.6873	1.6637	1.6847	1.6637	1.41	0.15	1.41
0.8	2.2041	2.1907	2.2174	2.1907	0.61	0.59	0.61
1.0	2.9086	2.8818	2.9022	2.8816	0.92	0.22	0.93

Table 3: Representing the values of post buckling load carrying capacity of clamped isotropic circular plates.

b_0/t	**Post buckling load carrying capacity (γ)**						
	PRESENT	REF [3]	REF [14]	REF [15]	Percentage error from REF [3]	Percentage error from REF [14]	Percentage error from REF [15]
0.0	1.0000	1.0000	1.0000	1.0000	0	0	0
0.2	1.0201	1.0210	1.0208	1.0205	0.08	0.06	0.03
0.4	1.0805	1.0840	1.0832	1.0835	0.32	0.24	0.27
0.6	1.1812	1.1893	1.1873	1.1887	0.68	0.51	0.63
0.8	1.3221	1.3373	1.3330	1.3366	1.13	0.81	1.08

1.0	1.5032	1.5286	1.5203	1.5278	1.66	1.12	1.61

The following graphs demonstrate a comparative study of the thermal post buckling load carrying capacity of circular plates for various b_0/t values with simply supported and clamped boundary conditions. From the figures, it is clear that the numerical results in the present study, along with those obtained from the references give more or less the same values of γ for simply supported and clamped boundary conditions.

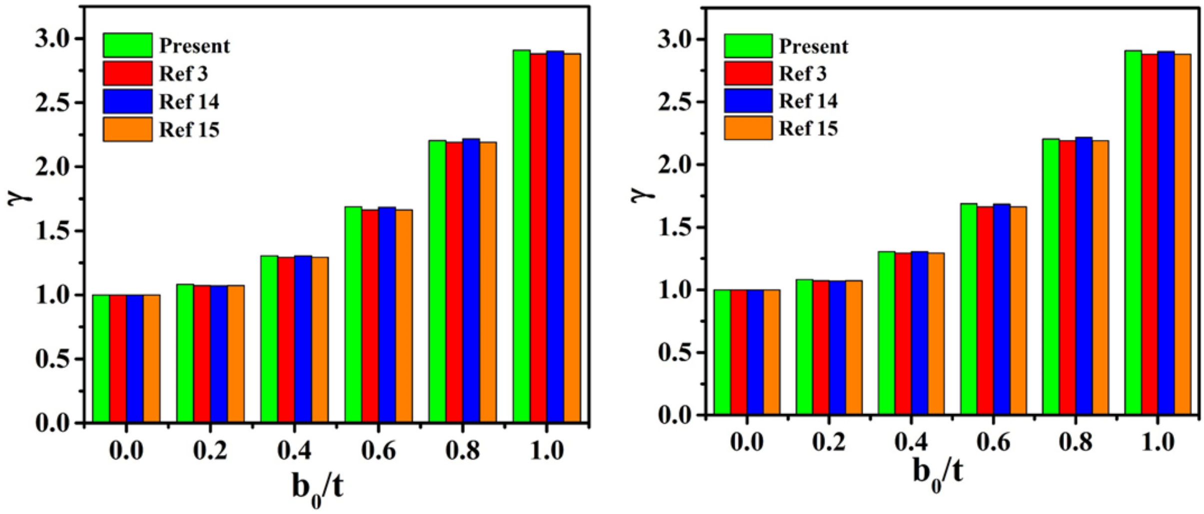

Figure 1: The bar graph comparing the present numerical results with the standard reference literature values for simply supported and clamped boundary conditions.

4. Conclusions

A comparative study on the thermal post buckling behavior of circular plates with both simply supported and clamped boundary conditions are compiled. The radial edge tensile load and hence the thermal post buckling load carrying capacity of the circular plates are evaluated by using a mathematical formulation based on Von Karman nonlinearities; the results obtained from the different known literatures are compared. The numerical results obtained from the present formulation minimizes the error reported in the previous studies and the calculated post buckling load carrying capacity of circular plates fits well with the values from the literatures. The present mathematical formulation can be used as a simple numerical method in calculating the post-buckling load of circular plates.

Acknowledgement

The authors would like to thank the Universiti Malaysia Pahang and Ministry of Higher Education, Malaysia for the provision in terms of financial form from the research grants (RDU160330).

Nomenclature

a = radius of the circular plate
N_r = uniform radial edge compressive load per unit length
$N_{r_{cr}}$ = linear buckling load
$N_{r_{NL}}$ = total uniform radial edge compressive load per unit length
N_{r_T} = uniform radial edge tensile load per unit length due to large lateral displacements

T = thickness of the plate
$\varepsilon_r, \varepsilon_\theta$ = in – plane strains
ν = Poisson's ratio
U = strain energy
r, θ = radial and circumferential coordinates
W = work done
χ_r, χ_θ = curvatures
r_1, r_2 = internal and external radii
α_1 - α_6 = generalized displacements
β = orthotropic parameter
E_r, E_θ = Young's moduli in radial and circumferential directions
λ = post buckling load parameter
b_0 = central (maximum) lateral displacement of the circular plate

References

1. Friedrichs, K.O. and J. Stoker, *Buckling of the circular plate beyond the critical thrust*, 1941.
2. Bodner, S.R., *The post buckling behavior of a clamped circular plate,* Quarterly of Applied Mathematics, 1955, **12**(4): p. 397-401.
3. Raju, K.K. and G.V. Rao, *Thermal post-buckling of circular plates,* Computers & Structures, 1984, **18**(6): p. 1179-1182.
4. Raju, K.K. and G.V. Rao, *Post-buckling of cylindrically orthotropic linearly tapered circular plates by finite element method,* Computers & Structures, 1985, **21**(5): p. 969-972.
5. Rao, G.V. and K.K. Raju, *Post-buckling behaviour of elastic circular plates using a simple finite element formulation,* Computers & Structures, 1979, **10**(6): p. 911-913.
6. Rao, G.V. and K.K. Raju, *A reinvestigation of post-buckling behaviour of elastic circular plates using a simple finite element formulation,* Computers & Structures, 1983, **17**(2): p. 233-235.
7. Rao, G.V., K.K. Raju, and N.R. Naidu, *Post-buckling behaviour of circular plates with elastically restrained edges subjected to thermal loads,* Computers & Structures, 1992, **45**(1): p. 209-210.
8. Rao, G.V. and R.R. Varma, *Simple formulation to predict thermal postbuckling load of circular plates,* AIAA Journal, 2007, **45**(7): p. 1784-1786.
9. Varma, R.R. and G.V. Rao, *Novel Formulation to Study Thermal Postbuckling of Circular Plates with Edges Elastically Restrained against Rotation,* Journal of Engineering Mechanics, 2011, **137**(10): p. 708-711.
10. Venkateswara Rao, G., et al., *Solution for large amplitude vibrations of circular plates via modified Berger's approximation,* The IES Journal Part A: Civil & Structural Engineering, 2012, **5**(1): p. 56-61.
11. Kanaka Raju, K. and G. Venkatewara Rao, *Finite-element analysis of post-buckling behaviour of cylindrically orthotropic circular plates,* Fibre Science and Technology, 1983, **19**(2): p. 145-154.
12. Yamaki, N., *Influence of large amplitudes on flexural vibrations of elastic plates,* ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 1961, **41**(12): p. 501-510.
13. Berger, H.M., *A new approach to the analysis of large deflections of plates,* 1954.
14. Ramaraju, R.V. and V.R. Gundabathula, *Reinvestigation of intuitive approach for thermal postbuckling of circular plates,* AIAA Journal, 2009, **47**(10): p. 2493-2495.
15. Li, S.-R., R.C. Batra, and L.-S. Ma, *Vibration of Thermally Post-Buckled Orthotropic Circular Plates.* Journal of Thermal Stresses, 2007. **30**(1): p. 43-57.