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On circular m-consecutive-k,l-out-of-n:F systems

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Abstract. A general class of circular systems, called circular m-consecutive-k, l-out-of-n:F systems, is considered. For a system with independent and identically distributed component lifetimes, characteristics of the system's lifetime are studied via signature-based mixture representations. In deriving results combinatorial arguments are used.

1. Introduction

Since Kontoleon [1] introduced and studied the (linear) consecutive-k-out-of-n:F systems, many papers have appeared on consecutive-type systems. This is due to the fact that such systems have been used to model communication networks, oil pipeline systems, quality control inspection procedures, radar detection systems, etc. A linear (circular) consecutive-k-out-of-n:F system consists of n components ordered linearly (circularly) which fails if and only if there are k consecutive failed components. The circular system was introduced by Derman et al. [2]. Griffith [3] introduced and studied the m-consecutive-k-out-of-n:F system which is a system consisting of n components ordered linearly and fails if and only if there are m non-overlapping runs of k consecutive failed components. Alevizos et al. [4] considered the circular m-consecutive-k-out-of-n:F model for a system with circularly ordered components. Agarwal and Mohan [5] proposed and studied the m-consecutive-k-out-of-n:F system with overlapping runs. This system consists of n linearly ordered components and fails if and only if there are at least m overlapping runs of k consecutive failures. Eryilmaz and Mahmoud [6] introduced and studied the linear m-consecutive-k, l-out-of-n:F (m-lin/con/$(k, l)/n$:F) system that fails if and only if there are at least m l-overlapping runs of k consecutive failed components. The number of l-overlapping failure runs of length k is the number of failure runs of length k each of which may have overlapping (common) part of length at most l with the previous failure run of length k that has been enumerated.

The consecutive systems that have been studied are assumed to have components which function independently with the same probability, independently with different probabilities, dependently in a Markovian fashion or they are exchangeable dependent. Surveys on the subject can be found in Chao et al. [7], Kuo and Zuo [8], Eryilmaz [9] and Levitin [10].

In the paper, we study the circular m-consecutive-k, l-out-of-n:F system (m-cir/con/$(k, l)/n$:F), i.e., a system consisting of n circularly ordered components that fails if and only if there are at least m, $m \geq 1$, l-overlapping runs of k, $0 \leq l < k \leq n$, consecutive failed components. This system, as the corresponding linear system, via its two flexible parameters m and l, is on the one hand a generalization of all the above mentioned circular consecutive systems and on the other hand models new types of circular systems. For $m = 1$ it reduces to a circular consecutive-k-out-of-n:F system and for $l = 0$ it corresponds to a circular m-consecutive-k-out-of-n:F system.

m-cir/con/$(k,l)/n$:F was introduced by Makri and Psillakis [11] and studied using a run counting statistic $X^{(C)}_{n;k,l}$, $0 \leq l < k \leq n$, denoting the number of l-overlapping runs of failures of length k in a sequence $\{X_i\}_{i=1}^n$ of n two-state trials (failure-success) ordered on a circle. These authors also provided a potential application concerning an alarm or supervision system for a circular accelerator or for the core of a nuclear plant based on a m-cir/con/$(k,l)/n$:F system. The application takes into account the model's flexibility due to parameters m, l. The approach, instead, followed in the paper is based on the computation of mixture representations of the system lifetime distribution relied on certain sets of system's signatures. These mixtures have been proved to be useful tools in evaluating lifetime, and hence reliability, characteristics of a system and in comparing different systems, too.

In the signature-based mixture representations of a system consisting of n independent and identically distributed (i.i.d.), or more generally of exchangeable, component lifetimes crucial role has the vector $\boldsymbol{r}(n) = (r_1(n), r_2(n), \dots, r_n(n))$. In a n-component system, $r_i(n)$, $i = 1,2,\dots,n$, denotes the number of path sets of the structure with exactly i (working) components, and its computation can be relied on combinatorial arguments. Using these numbers, one can find explicitly signature-based representations (i.e. signatures, minimal and maximal signatures) for the system's structure and consequently, he/she can evaluate lifetime characteristics of the system. The theory on signature-based mixture representations of coherent systems has been developed for i.i.d. component lifetime distributions (see Samaniego [12], Navarro et al. [13] and the references therein) and is extended to exchangeable sequences by Navarro and Rychlik [14]. Further notable works on signatures are, among others, those of Boland [15], Eryilmaz [16], Eryilmaz et al. [17], Triantafyllou and Koutras [18], Koutras et al. [19] and Eryilmaz et al. [20].

In the paper, the lifetimes of system components are assumed to be i.i.d. In Section 2, we present some necessary preliminaries on signature-based mixture representations for the survival function (reliability) of a coherent system. These representations are then used to obtain results, presented in Section 3, on circular m-consecutive-k,l-out-of-n:F systems. Numerical examples illustrate further the theoretical results.

2. Signature-based mixture representations

The computation of system signatures is a counting problem and in essence, a combinatorial one (Samaniego [12]). To be more specific, let $T_1, T_2, \dots, T_n$, be the lifetimes of the components of a n-component coherent system that has lifetime T. Let us assume that $T_1, T_2, \dots, T_n$ are i.i.d. random variables from a continuous distribution, say F, on $(0,\infty)$. The signature of the system is defined as the vector $\boldsymbol{s} = \boldsymbol{s}(n) = (s_1, s_2, \dots, s_n)$ with

$$s_i = s_i(n) = P(T = T_{i:n}), \quad i = 1,2,\dots,n, \tag{1}$$

where $T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n}$ are the order statistics of the lifetimes $T_1, T_2, \dots, T_n$. $T_{i:n}$ represents the lifetime of an i-out-of-n failure system (i/n:F) that is, a system for which the i-th component failure causes its failure. Obviously, the signature $\boldsymbol{s}$ of a system is a probability vector, i.e. $s_i \geq 0, \sum_{i=1}^n s_i = 1$, since $P(T \in \{T_{1:n}, T_{2:n}, \dots, T_{n:n}\}) = 1$. As a counting problem the evaluation of the signature of a coherent system can be defined via (Kochar et al. [21])

$$s_i = n_i/n!\,, i = 1,2,\dots,n, \tag{2}$$

where n_i is the number of orderings of the component lifetimes for which the ith component failure causes system failure. That is, s_i represents the ratio of n_i to the (total) number of possible orderings, $n!$, of the n failure times $T_1, T_2, \dots, T_n$. Alternatively, it holds (Boland [15])

$$s_i(n) = \alpha_{n-i+1}(n) - \alpha_{n-i}(n) = \frac{r_{n-i+1}(n)}{\binom{n}{n-i+1}} - \frac{r_{n-i}(n)}{\binom{n}{n-i}}, 1 \leq i \leq n, \tag{3}$$

where $\alpha_i = \alpha_i(n) = r_i(n)/\binom{n}{i}$, $i = 1,2,\dots,n$ ($\alpha_0(n)$=0, convention). $r_i = r_i(n)$, is the number of path sets of a $n-$component system with exactly i (working) components, and consequently α_i represents the proportion of them among $\binom{n}{i}$ such possible sets. Formula (3) is a fundamental tool for the computation of the signature of a coherent system since combinatorial arguments concerning the structure of the system can be used to derive the number of path sets of the system with i components.

Specifically, r_i can be determined by calculating the (total) number of binary sequences satisfying certain conditions which depend on the structure of the under study system.
Given the system's signature, the distribution of the system lifetime T, for i.i.d. component lifetimes with a distribution $F(\mathrm{t}) = \mathrm{P}(T_i \leq t), i = 1,2, \ldots, n,$ can be provided (Samaniego [12]) by

$$\bar{F}_T(t) = P(T > t) = \sum_{i=1}^{n} s_i\, P(T_{i:n} > t) = \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} \binom{n}{j} (F(t))^j (1 - F(t))^{n-j}. \tag{4}$$

Accordingly, the mean lifetime of the system can be expressed via its signature as

$$E(T) = \sum_{i=1}^{n} s_i\, E(T_{i:n}). \tag{5}$$

System signature is a useful tool in comparing systems in terms of stochastic ordering (see, Kochar et al. [21], Samaniego [12]). In brief, let $\boldsymbol{s}^{(u)} = (s_1^{(u)}, s_2^{(u)}, \ldots, s_n^{(u)}), u = 1,2,$ be the signatures of two coherent systems with respective lifetimes $T^{(u)}, u = 1,2,$ both based on n i.i.d. components with a common distribution F. Then,

$$\boldsymbol{s}^{(1)} \leq_{st} \boldsymbol{s}^{(2)} \text{ iff } \sum_{j=i}^{n} s_j^{(1)} \leq \sum_{j=i}^{n} s_j^{(2)}, i = 1,2, \ldots, n \tag{6}$$

and

$$\text{if } \boldsymbol{s}^{(1)} \leq_{st} \boldsymbol{s}^{(2)} \text{ then } T^{(1)} \leq_{st} T^{(2)}, \tag{7}$$

that is, $T^{(1)}$ is stochastically smaller than $T^{(2)}$, since the assumption $\boldsymbol{s}^{(1)} \leq_{st} \boldsymbol{s}^{(2)}$, implies

$$\bar{F}_{T^{(1)}}(t) = P\big(T^{(1)} > t\big) \leq P\big(T^{(2)} > t\big) = \bar{F}_{T^{(2)}}(t), \tag{8}$$

for all t, i.e. $T^{(1)}$ is less likely than $T^{(2)}$ to take values beyond t.
Alternatively, the survival function $\bar{F}_T(t)$ or the reliability function (polynomial for i.i.d. components) of the system can be written (since $\alpha_j = \sum_{i=n-j+1}^{n} s_i, j = 1,2, \ldots, n$) as

$$R\big(p(t)\big) = P(T > t) = \sum_{j=1}^{n} \alpha_j \binom{n}{j} p(t)^j q(t)^{n-j}, \tag{9}$$

where $p(t) = \bar{F}(t) = 1 - F(t) = 1 - q(t)$.

Furthermore, Eq. (4) can also be written as (see Appendix)

$$\bar{F}_T(t) = 1 - \sum_{i=1}^{n} s_i I_{F(t)}(i, n - i + 1), \tag{10}$$

where $I_x(v, \mu)$ is the incomplete Beta function ratio, given by $I_x(v, \mu) = \frac{\Gamma(v+\mu)}{\Gamma(v)\Gamma(\mu)} B(v; \mu; x)$ and $B(v; \mu; x) = \int_0^x t^{v-1} (1 - t)^{\mu-1}$, $0 \leq x < 1$, is the incomplete Beta function.

For n_f and n_w denoting the minimum number of failed components that may cause system failure and the maximum number of failed components such that the system can still work, respectively, it is true that

$$R(p) = \sum_{i=n-n_w}^{n} r_i(n) p^i (1 - p)^{n-i} = 1 - \sum_{i=n_f}^{n_w+1} s_i(n)\, I_{1-p}(i, n - i + 1), \tag{11}$$

where $p = \bar{F}(t_0)$, for a fixed time t_0, and

$$E(T) = \sum_{i=n_f}^{n_w+1} s_i(n) E(T_{i:n}). \tag{12}$$

The numbers n_f and n_w depend on the under study system.
Navarro et al. [13] proved that any coherent system could be written as a generalized mixture of series or parallel systems. More specifically,

$$\bar{F}_T(t) = P(T > t) = \sum_{i=1}^{n} a_i P(T_{1:i} > t) = \sum_{i=1}^{n} \beta_i P(T_{i:i} > t), \tag{13}$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ are vectors, satisfying the conditions $\sum_{i=1}^{n} a_i = 1$, $\sum_{i=1}^{n} \beta_i = 1$ and are called, respectively, minimal and maximal signature of the system. Eryilmaz [16]

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proved that the minimal and maximal signature of any coherent system with exchangeable (and consequently with i.i.d.) components can be given as

$$a_i = \sum_{j=n-i}^{n_w} (-1)^{j-(n-i)} \binom{j}{n-i} r_{n-j}(n), \tag{14}$$

$$\beta_i = -\sum_{j=0}^{\min(i,n_w)} (-1)^{i-j} \binom{n-j}{n-i} r_{n-j}(n). \tag{15}$$

In (13), $T_{1:i}$ and $T_{i:i}$ are the lifetimes of an i-component series and a parallel system, respectively. Since,

$$P(T_{1:i} > t) = P(T_1 > t, T_2 > t, \dots, T_i > t) = (\bar{F}(t))^i \tag{16}$$

and

$$P(T_{i:i} > t) = 1 - P(T_1 \le t, T_2 \le t, \dots, T_i \le t) = 1 - (1 - \bar{F}(t))^i, \tag{17}$$

knowing the common component survival function $\bar{F}(t)$ and the number of path sets $r_i(n)$ the only that remains to determine the system's survival function $\bar{F}_T(t)$ is to specify n_w for the under study structure.

3. Circular m-consecutive-k, l-out-of-n:F systems

Lemma 1 (Makri et al. [22]). The number of allocations of γ indistinguishable balls into r distinguishable cells, i specified of which have capacity $m-1$ and each of the rest $r-i$ has capacity $n-1$ is given by

$$C_{i,r-i}(\gamma; m-1, n-1) = \sum_{j_1=0}^{\left\lfloor \frac{\gamma}{m} \right\rfloor} \binom{i}{j_1} \sum_{j_2=0}^{\left\lfloor \frac{\gamma - m j_1}{n} \right\rfloor} (-1)^{j_1+j_2} \binom{r-i}{j_2} \binom{\gamma - m j_1 - n j_2 + r - 1}{r-1}, \tag{18}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Corollary 1 (Makri et al. [22]). Let $C_{i,r-i}(\gamma; m-1, n-1)$ be as in Lemma 1. Then,

$$C_{i,r-i}(\gamma; m-1, m-1) \equiv C(\gamma, r; m-1) = \sum_{j=0}^{\left\lfloor \frac{\gamma}{m} \right\rfloor} (-1)^j \binom{r}{j} \binom{\gamma - mj + r - 1}{r-1}. \tag{19}$$

The proofs of Theorem 1 and Proposition 1 are given in Appendix.

Theorem 1. The number, $r_i(n)$, of path sets with i elements of a circular m-consecutive-k, l-out-of-n:F system is given by the formula

$$r_i(n) = \sum_{x=0}^{m-1} r_{i,x}(n) \tag{20}$$

(convention: $r_0(n) = 0$) where

$$r_{i,0}(n) = \frac{n}{i} C(n-i, i; k-1) \tag{21}$$

and for $x \ge 1$,

$$r_{i,x}(n) = \frac{n}{i} \sum_{j=1}^{i} \binom{i}{j} \binom{x-1}{j-1} C_{j,i-j}(n - i - jk - (x-j)(k-l); k-l-1, k-1). \tag{22}$$

Proposition 1. For a circular m-consecutive-k, l-out-of-n:F system, it is true that for $m(k-l) \le n \le m(k-l) + l$, $n_f = n$, $n_w = n-1$ and for $n \ge m(k-l) + l + 1$,

$$n_f = m(k-l) + l, \tag{23}$$

$$n_w = n - 2 - \left\lfloor \frac{n-1-n_f}{k} \right\rfloor. \tag{24}$$

Example 1. In Table 1, we depict the reliability $R_{k,m;n}^{(l)}(p)$, for $p = 0.50$ to 0.95, with step 0.05, of circular m-consecutive-k, l-out-of-n:F systems for $l = 0,1,2$ and $m = 2$, $k = 3$, $n = 10$. $R_{k,m;n}^{(l)}(p)$ can be computed using Eq. (11) or (13). The $r_i^{(l)}$, $s_i^{(l)}$, $a_i^{(l)}$ and $\beta_i^{(l)}$ are presented in Table 2. From Table 1, we observe that $R_{k,m;n}^{(l)}(p)$ does not increase with increasing l and is bounded from above by that of $l = 0$ and below by that of $l = 2 = k-1$. This feature is a common one for a m −cir/con/(k,l)/n:F with fixed m, n, k, p and varying l, $l = 0,1,\dots,k-1$.

Table 1. RELIABILITY $R^{(l)}_{3,2;10}(p), l = 0,1,2,$ of $2-$cir/con/$(3,l)$/10:F for $p = 0.50(0.05)0.95$

p/l	0	1	2
0.50	0.8623	0.7939	0.6670
0.55	0.9163	0.8656	0.7591
0.60	0.9532	0.9187	0.8360
0.65	0.9764	0.9551	0.8962
0.70	0.9895	0.9779	0.9401
0.75	0.9961	0.9907	0.9693
0.80	0.9989	0.9968	0.9867
0.85	0.9998	0.9992	0.9956
0.90	1.0000	0.9999	0.9991
0.95	1.0000	1.0000	0.9999

Table 2. PATH NUMBERS, $r_i^{(l)}$, SIGNATURES, $s_i^{(l)}$, MINIMAL, $a_i^{(l)}$ AND MAXIMAL, $\beta_i^{(l)}$ SIGNATURES, OF 2-CIR/CON/$(3,l)/10$: F SYSTEMS FOR $l = 0,1,2$

i	$r_i^{(l)}$	$a_i^{(l)}$	$\beta_i^{(l)}$	$s_i^{(l)}$	$r_i^{(l)}$	$a_i^{(l)}$	$\beta_i^{(l)}$	$s_i^{(l)}$	$r_i^{(l)}$	$a_i^{(l)}$	$\beta_i^{(l)}$	$s_i^{(l)}$
	$l = 0$				$l = 1$				$l = 2$			
1	0	0	0	0.0000	0	0	0	0.0000	0	0	0	0.0000
2	0	0	0	0.0000	0	0	0	0.0000	0	0	0	0.0000
3	59	60	0	0.0000	29	30	0	0.0000	9	10	0	0.0000
4	185	-235	0	0.0000	155	-55	0	0.0000	95	25	10	0.0476
5	251	402	0	0.0000	241	-58	10	0.0397	201	-158	-10	0.1508
6	210	-375	25	0.1190	210	275	5	0.2222	200	265	15	0.3492
7	120	200	-40	0.3810	120	-350	-30	0.4881	120	-210	-50	0.3691
8	45	-60	15	0.5000	45	220	5	0.2500	45	80	55	0.0833
9	10	10	0	0.0000	10	-70	20	0.0000	10	-10	-20	0.0000
10	1	-1	1	0.0000	1	9	-9	0.0000	1	-1	1	0.0000

In Table 2 we present, respectively, the vectors of the path numbers $\boldsymbol{r}^{(l)} = \left(r_1^{(l)}, r_2^{(l)}, \dots, r_n^{(l)}\right)$, the signatures $\boldsymbol{s}^{(l)} = \left(s_1^{(l)}, s_2^{(l)}, \dots, s_n^{(l)}\right)$, the minimal and the maximal $\boldsymbol{a}^{(l)} = \left(a_1^{(l)}, a_2^{(l)}, \dots, a_n^{(l)}\right)$ and $\boldsymbol{\beta}^{(l)} = \left(\beta_1^{(l)}, \beta_2^{(l)}, \dots, \beta_n^{(l)}\right)$, signatures, $0 \le l \le k-1$, of circular m-consecutive-k, l-out-of-n:F systems for $m = 2, k = 3, n = 10$. By the $s_i^{(l)}$ columns of Table 2 we easily verify, using Eq. (6), that $\boldsymbol{s}^{(2)} \le_{st} \boldsymbol{s}^{(1)} \le_{st} \boldsymbol{s}^{(0)}$. Therefore, for $T^{(l)}_{k,m;n}$ denoting the lifetime of a m-cir/con/(k,l)/n:F system, it holds $T^{(2)}_{3,2;10} \le_{st} T^{(1)}_{3,2;10} \le_{st} T^{(0)}_{3,2;10}$, which is consistent with the findings of Table 1, i.e. $R^{(2)}_{3,2;10}(p) \le R^{(1)}_{3,2;10}(p) \le R^{(0)}_{3,2;10}(p)$.

Example 2. Let us consider that the component lifetimes $T_1, T_2, \dots, T_n$ are independent random variables with a common exponential distribution $F(t) = 1 - e^{-\lambda t}, t \ge 0$, and mean $1/\lambda, \lambda > 0$.

Then, the expected value of $T_{i:n}$ (i.e. of the i-th smallest component lifetime) is given by

$$E(T_{i:n}) = \lambda^{-1} \sum_{j=1}^{i} (n - j + 1)^{-1}, i = 1,2,\dots,n. \tag{25}$$

In Table 3, we compute, via Eq. (13), the survival function $\bar{F}^{(l)}_{T_{3,2;10}}(t)$, i.e. the reliability $R^{(l)}_{3,2;10}(t)$, of 2-cir/con/$(3,l)/10$:F, for $l = 0,1,2$ and several values of t, when $\lambda = 1$. The $a_i^{(l)}$ and $\beta_i^{(l)}$ are presented in Table 2.

In Table 4, we compute, by Eqs. (12) and (25), the mean lifetime, $E(T^{(l)}_{k,m;n})$, i.e. the mean time to failure (MTTF), of circular m-consecutive-k, l-out-of-n:F systems, for several values of n, m, k and

$0 \leq l \leq k-1$, when $\lambda = 1$. From the Table, we observe that $E(T_{k,m;n}^{(l)})$ is increasing in m and k and decreasing in n and l.

Table 3. SURVIVAL FUNCTION $\bar{F}_{T_{3,2;10}}^{(l)}(t), l = 0,1,2$ OF 2-CIR/CON/$(3, l)/10$:F SYSTEMS, FOR EXPONENTIAL MODEL WITH $\lambda = 1$

$t\backslash l$	0	1	2
0.25	0.9980	0.9948	0.9806
0.50	0.9569	0.9243	0.8448
0.75	0.8242	0.7463	0.6108
1.00	0.6256	0.5235	0.3821
1.50	0.2652	0.1904	0.1131
2.00	0.0860	0.0547	0.0275
2.50	0.0239	0.0140	0.0062
3.00	0.0061	0.0034	0.0013

Table 4. MTTF, $E(T_{k,m;n}^{(l)}), l = 0,1,\ldots,k-1$, OF m-CIR/CON/$(k,l)/n$:F SYSTEMS

n	m	k	l	MTTF	n	m	k	l	MTTF
10	2	3	0	1.2325	20	3	3	0	1.0525
			1	1.1056				1	0.9381
			2	0.9389				2	0.7825
10	3	3	0	1.9290	20	2	4	0	1.1062
			1	1.5679				1	1.0565
			2	1.1512				2	0.9883
10	2	4	0	1.7623				3	0.8861
			1	1.6234					
			2	1.4448					
			3	1.2067					

4. Appendix

Proof of Equation (10). Since,

$$\sum_{j=i}^{n} \binom{n}{j} \left(F(t)\right)^j (1-F(t))^{n-j} = \frac{n!}{(i-1)!\,(n-i)!} \int_0^{F(t)} u^{i-1} (1-u)^{(n-i+1)-1} du$$

$$= I_{F(t)}(i, n-i+1)$$

and $P(T > t) = 1 - \sum_{i=1}^{n} s_i \sum_{j=i}^{n} \binom{n}{j} (F(t))^j (\bar{F}(t))^{n-j}$, the result follows.

Proof of Theorem 1. Let $r_{i,x}(n)$ denote the number of circularly ordered binary sequences of n trials with i successes (working components) and x l-overlapping failure (failed component) runs of length k. Then, it is clear that $r_i(n) = r_{i,0}(n) + r_{i,1}(n) + \cdots + r_{i,m-1}(n)$. We will consider the evaluation of $r_{i,x}(n)$ mainly as a problem of allocation of balls into cells. The i working components displayed on the circle form i cells with a cell defined between two working components. These cells are made distinguishable by labelling them. The number of allocations of $n-i$ failures (failed components) into i cells so that no cell receives more than $k-1$ failures is $C(n-i, i; k-1)$, by Corollary 1, so that

$r_{i,0}(n) = \frac{n}{i} C(n-i, i; k-1)$, since the sequences are circularly arranged and each of the $C(n-i, i; k-1)$ arrangements gives n arrangements by rotation and the set of $nC(n-i, i; k-1)$

arrangements is partitioned into sets of i like arrangements. For $x \geq 1$ we observe that the number of allocations of $n-i$ failures into i distinguishable cells (created by the i working components) with j of them receiving x l-overlapping failure runs of length k and no one of the remaining $i-j$ cells receiving more than $k-1$ failures is $\binom{x-1}{j-1} C_{j,i-j}(n-i-jk-(x-j)(k-l); k-l-1, k-1)$. But the j cells (the ones containing the l-overlapping failure runs of length k) can be chosen in $\binom{i}{j}$ ways, $j = 1,2,\dots, min\left\{i, \left\lfloor \frac{n-i}{k} \right\rfloor\right\}$. So, the total number of allocations of the $n-i$ failed components in the i cells yielding x l-overlapping failure runs of length k is $\sum_{j=1}^{i}\binom{i}{j}\binom{x-1}{j-1} C_{j,i-j}(n-i-jk-(x-j)(k-l); k-l-1, k-1)$. Since the created sequences are circular, using the same arguments as above, the result for $r_{i,x}(n)$, $x \geq 1$, follows.

Proof of Proposition 1. For $n \geq l+m(k-l)+1$, it is clear that $l+m(k-l)$ consecutive failures (Fs) may cause system's failure while $l+m(k-l)-1$ can not, so that $n_f = l+m(k-l)$. Working in a similar way with that of Eryilmaz and Mahmoud [6] we consider one of the positions in the circular sequence as the first one and going clockwise we put $x = l+m(k-l)-1$ failures followed and preceded by a success (S), i.e., we have a sequence of the form

$$\frac{S}{n}\frac{F}{1}\frac{F}{2}\cdots\frac{F}{x}\frac{S}{x+1}\overbrace{\cdots\cdots}^{n-x-2},$$

where the $n-x-2$ trials may include a maximum of $n-x-2-\left\lfloor \frac{n-x-2}{k} \right\rfloor$ failures so that the system can still work. Thus, $n_w = n-2-\left\lfloor \frac{n-l-m(k-l)-1}{k} \right\rfloor$. For $m(k-l) \leq n \leq l+m(k-l)$ the results for n_f and n_w are straightforward.

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