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A fuzzy mathematical model of West Java population with logistic growth model

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Abstract. In this paper we develop a mathematics model of population growth in the West Java Province Indonesia. The model takes the form as a logistic differential equation. We parameterize the model using several triples of data, and choose the best triple which has the smallest Mean Absolute Percentage Error (MAPE). The resulting model is able to predict the historical data with a high accuracy and it also able to predict the future of population number. Predicting the future population is among the important factors that affect the consideration is preparing a good management for the population. Several experiment are done to look at the effect of impreciseness in the data. This is done by considering a fuzzy initial value to the crisp model assuming that the model propagates the fuzziness of the independent variable to the dependent variable. We assume here a triangle fuzzy number representing the impreciseness in the data. We found that the fuzziness may disappear in the long-term. Other scenarios also investigated, such as the effect of fuzzy parameters to the crisp initial value of the population. The solution of the model is obtained numerically using the fourth-order Runge-Kutta scheme.

1. Introduction
Population growth is an important issue that a government in any country should pay attention. It is needed as a bases for population management action by the government, e.g. to make decision on how many new jobs should be provided to reduce the number of poor population. As an example, countries in West Africa which is a category of poor countries have an average annual population growth rate of 2,6% whereas ideally population growth rate is bellow1%[1]. This information can be used to either in providing new jobs or in reducing the growth rate of the populations.

In West Java, Indonesia, based on the result of the population census every ten years from 1980-2010 [2], the population is 23.434.003, 29.415.723, 35.723.473 and 43.053.732, respectively. It can be seen that the population of West Java Province every ten years has increased. The increasing number of population in a region causes many problems in various aspects, such as in social aspect, health aspects and environmental aspect. To overcome these problems, the government must prepare appropriate action for the future. One of them by knowing the projected population. The projection of the population of West Java Province can be done through mathematical modelling. Mathematical modelling is used here to estimate the population in the future, to know the population growth rate, and to know the maximum population limitation (the population carrying capacity).

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The term of population projection has seen discussed in several studies, e.g. by the authors in [1], [3], [4], [5], [6]. In this regard, we developed a mathematical model of population growth in West Java Province of Indonesia. The model used is the form of logistic differential equations. Sampling method in logistic model is done by varying sampling interval. The smallest MAPE interval becomes a consideration for projected population in West Java Province.

Furthermore, because the population census data may be containing uncertainty, we use fuzzy set theory with triangle membership function. This is done by considering the fuzzy initial value of the obtaining data, instead of the crispinitial value of the data. Hence the resulting mathematical model contains uncertainty in some sense.In this paper we investigate the effect of this fuzzy initial value to the population projection. We use numerical solution using the fourth-order Runge-Kutta method to develop the projection. Other scenario is also investigate, i.e. the effect of a fuzzy parameter on a crisp initial value of the population. In the following section we provide a brief introductory to some mathematical concept used in the subsequent discussion.

2. Preliminaries

This section briefly describes the main theoretical tools that will be used in developing the mathematical model for a population growth.

2.1. Logistic Growth Model

This model is called Verhulst model, a refinement of the exponential model and was first introduced by Pierre Verhulst in 1838 [5]. Verhulst points out that population growth depends not only on the size of the population but the extent of this size from its upper limit as carrying capacity. The logistic equation can be written as follow:

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right), \tag{1}$$

Where r represents the intrinsic growth rate, P represents the total population and K represents the carrying capacity. From equation (1) the following solution can be obtained:

$$P(t) = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-rt}}. \tag{2}$$

The values of parameters r and Kcan be estimated from the population $P(t)$when three data for three consecutive years with the same time-space are known. Let us assume that P_0be the population at $t=0$, $P_T$be the population at $t=T$ and P_{2T} be the population at $t=2T$, then from equation (2) it is obtained:

For $t=T$:

$$\frac{\beta}{\alpha}\left(1-e^{-\alpha T}\right) = \frac{1}{P_T} - \frac{e^{-\alpha T}}{P_0}. \tag{3}$$

For $t=2T$:

$$\frac{\beta}{\alpha}\left(1-e^{-2\alpha T}\right) = \frac{1}{P_{2T}} - \frac{e^{-2\alpha T}}{P_0}. \tag{4}$$

By dividing equation (4) by (3) to eliminate K, it is obtained:

$$e^{-\alpha T} = \frac{P_0(P_{2T}-P_T)}{P_{2T}(P_T-P_0)}, \tag{5}$$

so the population growth rate(r) and carrying capacity (K)can be written expicitly as:

$$r = \frac{1}{T}\ln\frac{P_{2T}(P_0-P_T)}{P_0(P_T-P_{2T})}, \tag{6}$$

$$K = \frac{P_T(P_0P_T-2P_0P_{2T}+P_TP_{2T})}{{P_T}^2-P_0P_{2T}}. \tag{7}$$

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These formulas is the main ingridient in the analysis of the data in the paper.

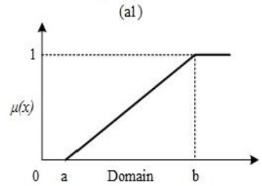
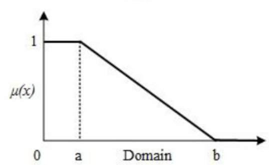
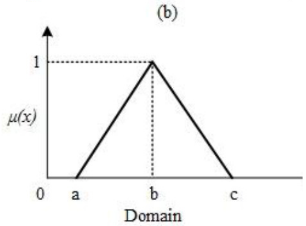
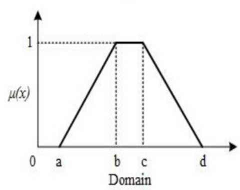
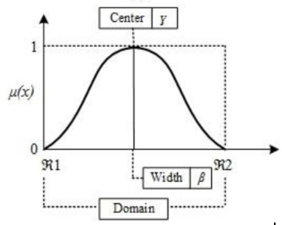
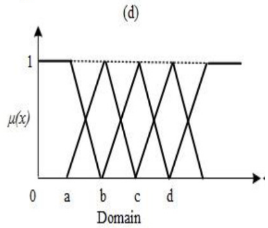
2.2 Mean Absolute Percentage Error (MAPE)

Mean Absolute Percentage Error (MAPE) is a statistical evaluation used to assess the suitability of a model in population projection. MAPE is expressed as a percentage. The concept of MAPE is very simple but becomes very important in choosing the best model from other models. A model with a smaller MAPE is a model chosen from other models. The mathematical from of MAPE is as follow:

$$MAPE = \frac{1}{N}\sum_{t=1}^{N}\left|\frac{x_t-\hat{x}_t}{x_t}\right| \times 100\% \tag{8}$$

where $x_t$represent the actual population at time t, $\hat{x}_t$ represents the population projection at time t and N is the number of observations of each population. The smallest MAPE value is the best because of the smaller percentage of error generated from a model. For the interpretation of MAPE value that if a MAPE value of less than 10% is very accurate projection, 10% to 20% is a very good projection, 21% to 50% is reasonable projection and over 51% is forecasting not accurate [7].

Table 1. Membership Function of Fuzzy Set Theory

Function	Membership Function	Function	Membership Function
Linear Representations (a1)  (a2) 	$\mu(x)=\begin{cases}0, x\le a\\ \frac{x-a}{b-a}, a\le x\le b\\ 1, x\ge b\end{cases}$ $\mu(x)=\begin{cases}1, x\le a\\ \frac{b-x}{b-a}, a\le x\le b\\ 0, x\ge b\end{cases}$	**Triangular Curve Representation** (b) 	$\mu(x) = \begin{cases}0, x\le a \; atau \; x\ge c\\ \frac{x-a}{b-a}, a\le x\le b\\ \frac{c-x}{c-b}, x\ge b\end{cases}$
Trapezoidal curve Representation (c) 	$\mu(x) = \begin{cases}0, x\le a \; atau \; x\ge c\\ \frac{x-a}{b-a}, a\le x\le b\\ 1, b\le x\le c\\ \frac{d-x}{d-c}, c\le x\le d\end{cases}$	**Gaussian Curve Representation** (e) 	$G(x;\beta,\gamma) = e^{-\frac{(x-\gamma)^2}{2\beta^2}}$
Combination Curve Representation (d) 	Combination of linear membership functions (up and down) and triangle membership functions.		

2.3 *Fuzzy Set Theory*

The fuzzy set theory is designed to represent the uncertainty, inaccuracies and vagueness of the data/information available. The fuzzy set is a generalization of the ordinary set concept. For the universe of discourse U, the fuzzy set is determined by the membership function that maps the member u to the membership range in the interval [0,1]. For the set of membership fuctional members of discrete value 0 and 1.Membership function is a curve showing the mapping of data input points into their membership values (often called membership degrees) that have intervals between 0 and 1. One way to get membership value is through a functional approach. Some forms of membership function are linear representations (top and bottom), triangular curve representation, triangle curve, trapezoidal curve and Gaussian curve representations, Combination Curve [9].The graph of membership function for each curve can be seen in the following Table 1. In this paper we only concentrate in the triangular fuzzy number, since other forms can be use analogously.

2.4 *The Fourth-Order Runge-Kutta Method*

The most commonly used Runge-Kutta method is the fourth-order classical Runge-Kutta method, along with the equation of the classical fourth-order Runge-Kutta method [8].

$$y_{i+1} = y_i + \left[\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\right] h, \tag{9}$$

with

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}hk_1\right), \\ k_3 &= f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}hk_2\right), \\ k_4 &= f(x_i + h,\ y_i + hk_3), \end{aligned}$$

where k is recursively computed, i.e. k_1 is needed in k_2 calculation, k_2 is needed in k_3 calculation and k_3 is needed in k_4 calculation.

3. Results and Discussion

To select the most appropriate projection in the Logistic model, the population projection of 12 different samples of three points are taken with the same spaces between the points , i.e. the sampling interval 1, 2, 3 up to interval 12. The resulting projection is then compared with the actual data. The best projection is the projection that produces the smallest MAPE. For each interval on sampling, the population data taken as much as three pieces. If P_0 is the population at $t = 0$, P_T is the population at $t = T$ and P_{2T} is the population at $t = 2T$, so for interval 1, data was taken in 1990 as P_0, 1991 as P_T and 1992 as P_{2T}. At interval 2, data was taken in 1990 as P_0, 1992 as P_T and 1994 as P_{2T}. Likewise until at interval 12.In the logistic model, interval 12 is chosen as the best interval because it has a projection close to the actual data. The following is the resulting growth rate and carrying capacity of interval 12 to represent the best growth rate and carrying capacity of the model, namely

$$r^* = 0.040718403, \quad K^* = 75692359.51, \quad \text{with } MAPE = 1.410076124. \tag{10}$$

Using the values of parameters in (10), the projection by the logistic equation is presented in Figure 1. The logistic solution is compared to the actual data. The figure shows the projection up to time t = 100 and the actual data is up to time t = 25. The projection from the logistic model is pretty close. This is clarified by the resulting MAPE from the projection solution to the actual data which is 1.410076124. In the next section we look for the numerical solution of the best model by using the fourth-order Runge-Kutta method for various scenario of the fuzziness of the initial values and the model parameters. We assume that initial value of the population is a triangular fuzzy number. Other scenario is we assume that carrying capacity (K) and growth rate (r) are also fuzzy numbers with triangular membership functions.

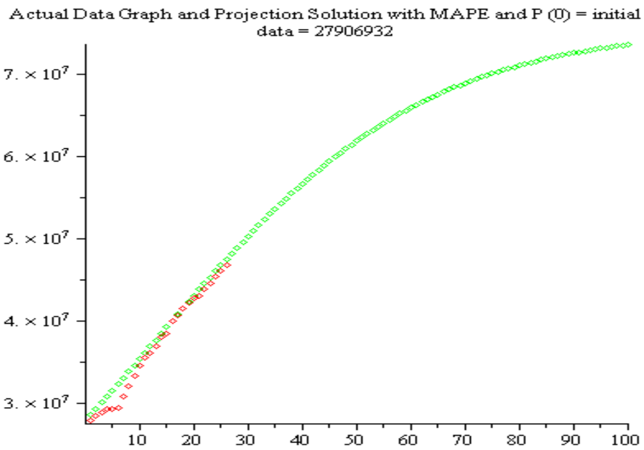


Figure 1. Graph of projection solution and actual data

3.1 Projection with the Fuzzy Initial Value

Suppose the initial value of the population (P_0) is a fuzzy number in the form of triangular membership $P_0 = (P_{0_1}, P_{0_2}, P_{0_3})$ with $P_{0_1} = 26906932, P_{0_2} = 27906932$ dan $P_{0_3} = 28906932$. Using the fourth-order Runge-Kutta method it is obtained the projection as in Figures 2, 3, 4 and 5.

(2a) (2b)

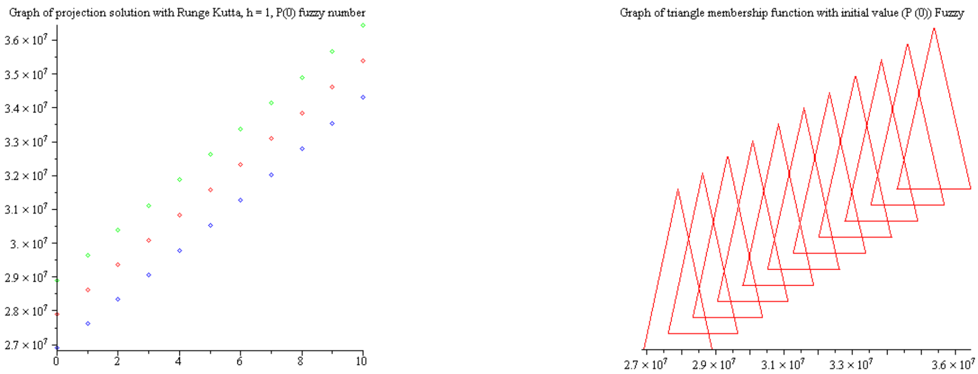


Figure 2. (2a) Numerical solution with the triangular fuzzy number with $P_{0_1} = 26906932, P_{0_2} = 27906932$ and $P_{0_3} = 28906932$ for $T = 10$; (2b) Graph of triangle membership function for the model solution.

In Figures 2 to 5, specifically figure (2a), (3a), (4a) and (5a) it is illustrated the numerical solution with the triangular fuzzy number P_0 with $P_{0_1} = 26906932, P_{0_2} = 27906932$ and $P_{0_3} = 28906932$ for projection up to 10, 100, 510 and 1000 iterations of time. The graph is increasing and grows asymptotically to the carrying capacity. Further, in Figures 2 to 5, specifically figure (2b), (3b), (4b) and (5b) it is illustrated the graph of triangle membership function for the model solution for projection up to 10, 100, 510 and 1000 iterations of time.

(3a) (3b) (3c)

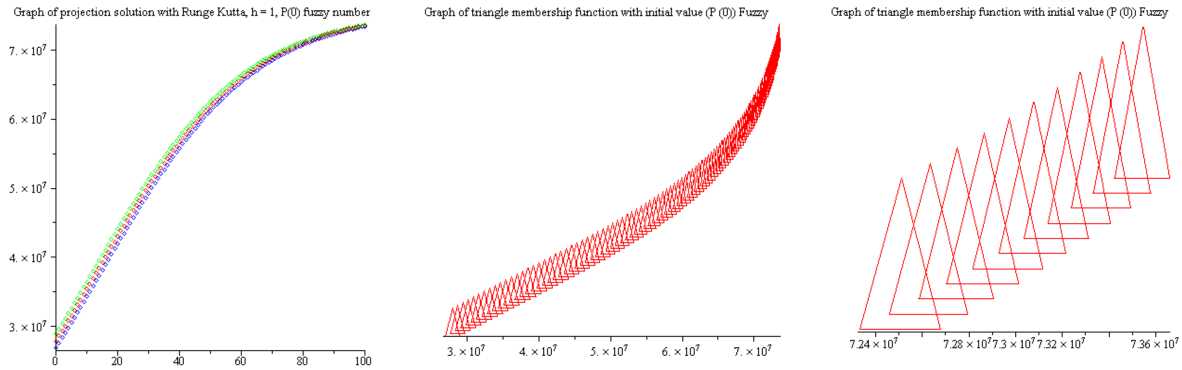


Figure 3. (3a) Numerical solution with the triangular fuzzy number with $P_{0_1} = 26906932, P_{0_2} = 27906932$ and $P_{0_3} = 28906932$ for $T = 100$; (3b) Graph of triangle membership function for the model solution with $T = 100$; (3c) Graph of triangle membership function for the model solution with $T = 90 \cdots 100$.

(4a) (4b) (4c)

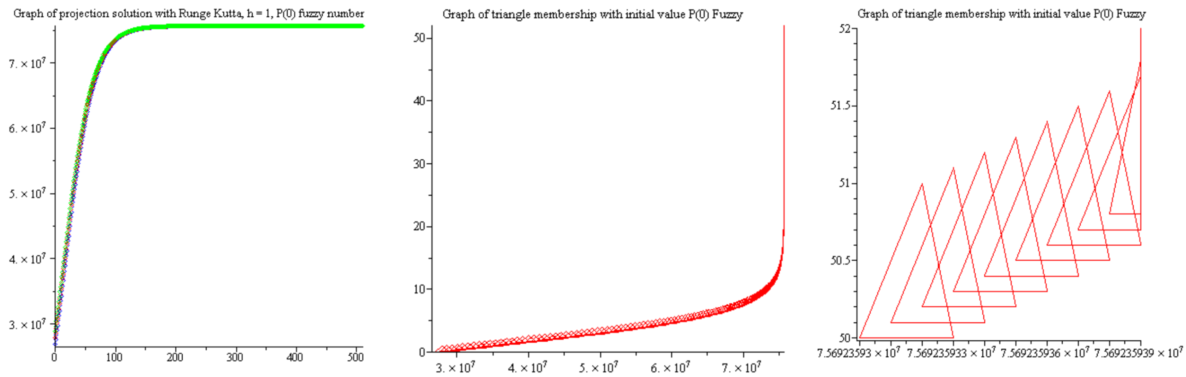


Figure 4. (4a) Numerical solution with the triangular fuzzy number with$P_{0_1} = 26906932, P_{0_2} = 27906932$ and $P_{0_3} = 28906932$ for $T = 510$; (4b) Graph of triangle membership function for the model solution with $T = 510$; (4c) Graph of triangle membership function for the model solution with $T = 500 \cdots 510$.

(5a) (5b) (5c)

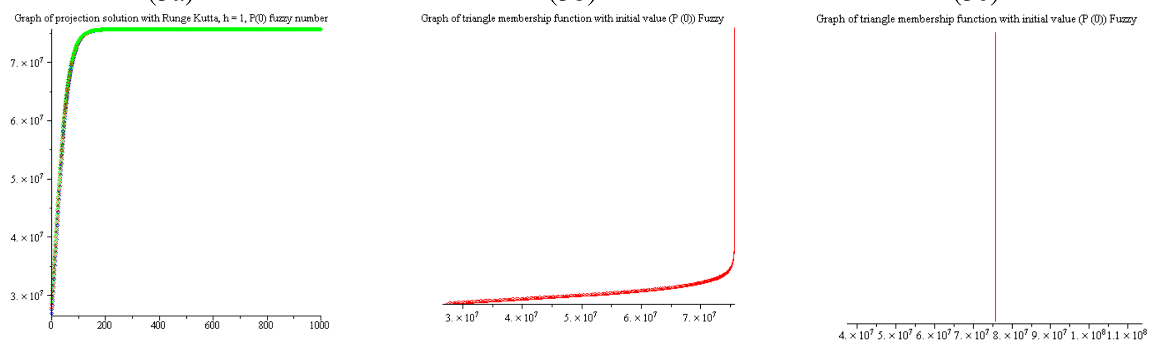


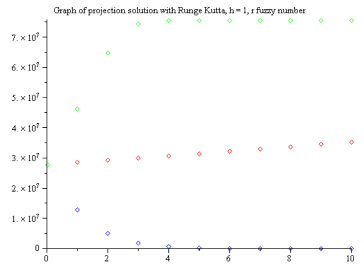
Figure 5. (5a) Numerical solution with the triangular fuzzy number with $P_{0_1} = 26906932, P_{0_2} = 27906932$ and $P_{0_3} = 28906932$ for $T = 1000$; (5b) Graph of triangle membership function for the model solution with $T = 1000$; (5c) Graph of triangle membership function for the model solution with $T = 990 \cdots 1000$.

We note that the support of the resulting triangular fuzzy number is getting smaller and smaller, until finally disappear and collapse to form a crisp number as indicated in Figure 4. This means that even though at first there is uncertainty in the initial value but in the long-term this uncertainty is dissapearing. To see that the resulting Runge-Kutta solution is indeed a fuzzy triangular number, we plot the solution at iteration 90 to 100 (Figure 3c). Figure (4c) is a graph of triangular membership function which in its 510 uncertainty is already disappearing, and at iteration 990 to 1000 the uncertainty is completely dissapear.

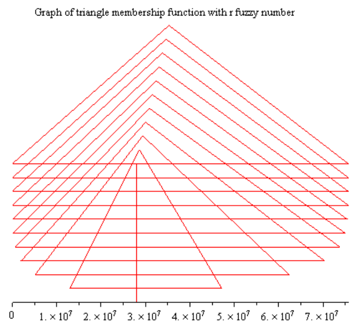
3.2 Projection with the Fuzzy Growth Rate (r)

In this section we assume that the parameters in the model is fuzzy. Suppose the growth rate of the population (r) is a fuzzy number in the form of triangular membership $r = (r_1, r_2, r_2)$ with $r_1 = -1.040718403, r_2 = 0.040718403$ and $r_3 = 1.040718403$. Using the fourth-order Runge-Kutta method it is obtained the projection as in Figure 6, 7 and 8. Figures 6 to 8, specifically figure (6a), (7a) and (8a) illustrate the numerical solution with the three initial values of the population given equally $P_{0_1} = 27906932$, $P_{0_2} = 27906932$ and $P_{0_3} = 27906932$ but r subjectively based on fuzzy set theory, i.e. $r_1 = -1.040718403, r_2 = 0.040718403$ and $r_3 = 1.040718403$ for projection up to 10, 100, and 500 iterations of time. The population increases when $r_2 = 0.040718403$ and $r_3 = 1.040718403$ grows asymptotically towards carrying capacity. Whereas when $r_1 = -1.040718403$ the population will go down and eventually extinct.

(6a)



(6b)



(6c)

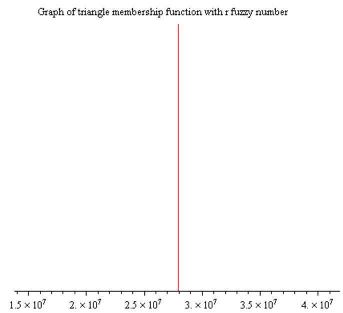
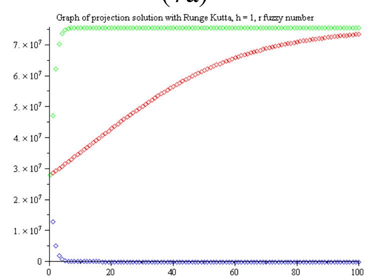
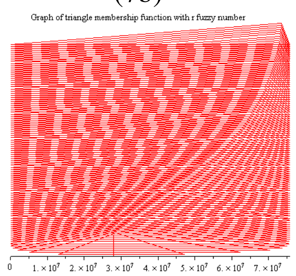


Figure 6. (6a) Numerical solution with the triangular fuzzy number with $r_1 = -1.040718403, r_2 = 0.040718403$ and $r_3 = 1.040718403$ for $T = 10$; (6b) Graph of triangle membership function for the model solution with $T = 10$; (6c) Graph of triangle membership function for the model solution with $T = 0$

(7a)



(7b)



(7c)

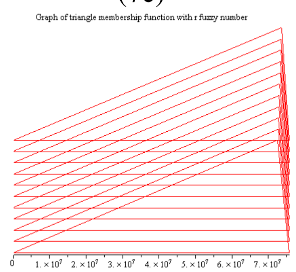


Figure 7. (7a) Numerical solution with the triangular fuzzy number with $r_1 = -1.040718403, r_2 = 0.040718403$ and $r_3 = 1.040718403$ $T = 100$; (7b) Graph of triangle membership function for the model solution $T = 100$; (7c) Graph of triangle membership function for the model solution $T = 90 - 100$

(8a) (8b) (8c)

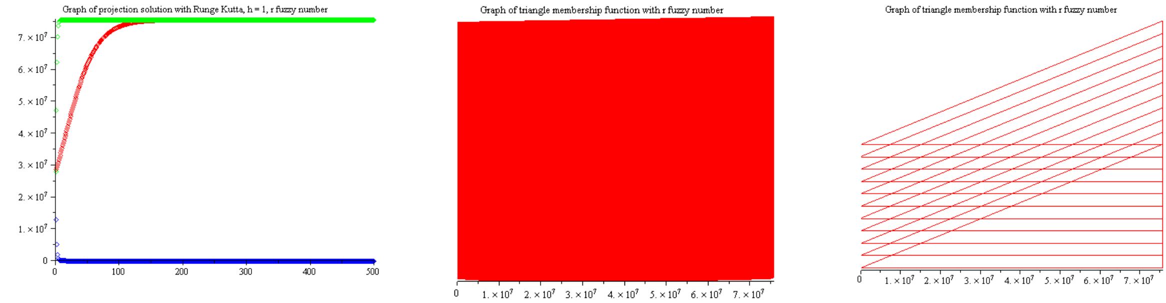


Figure 8. (8a) Numerical solution with the triangular fuzzy number with $r_1 = -1.040718403, r_2 = 0.040718403$ and $r_3 = 1.040718403$ for $T = 500$; (8b) Graph of triangle membership function for the model solution $T = 500$; (8c) Graph of triangle membership function for the model solution $T = 490 - 500$;

In Figures 6 to 8, specifically Figure (6b), (7b), and (8b) it is illustrated the graph of triangle membership function for the model solution for projection up to 10, 100 and 500 iterations of time. This figure shows the larger triangular membership function. In Figure (6c) it is shown a graph of triangle membership function at iteration 0, in which it is shown that is does not contain uncertainty on data. Figure (7b) is a graph of triangle membership function at iteration 0 to 100 and Figure (8b) is a graph of triangle membership function at iteration 0 to 500. In Figure (8b) is almost identical to Figure (7b), but because of the higher iteration so that the membership function in Figure (8b) coincides and looks like a blocked red. Figure (7c) is a graph of triangle membership function at iteration 90 to 100 and Figure (8c) is a graph of triangle membership function at iteration 490 to 500.

3.3 Projection with the Fuzzy Carrying Capacity (K)

Suppose the carrying capacity of the population (K) is a fuzzy number in the form of triangular membership $K = (K_1, K_2, K_2)$ with $K_1 = 74692359.51$, $K_2 = 75692359.51$ and $K_3 = 76692359.51$. Using the fourth-order Runge-Kutta method it is obtained the projection as in Figures 9, 10 and 11. In Figures 9 to 11, specifically figures (9a), (10a) and (11a), it is illustrated the numerical solution with the three initial values of the population given $P_{0_1} = 27906932, P_{0_2} = 27906932$ and $P_{0_3} = 27906932$ but K subjectively based on fuzzy set teory, i.e. $K_1 = 74692359.51$, $K_2 = 75692359.51$ and $K_3 = 76692359.51$ on iterations 10,100 and 500. Graphs 9, 10, 11 show that the population always rises and grows asymptotically to 3 different carrying capacities.

(9a) (9b) (9c)

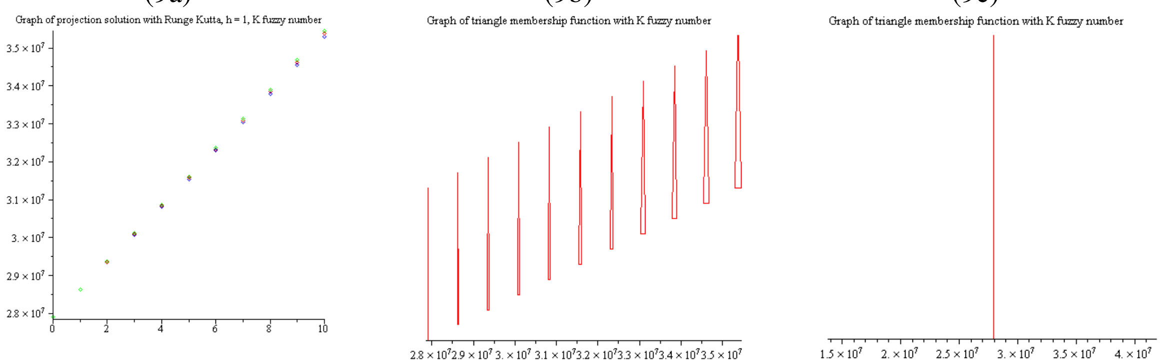


Figure 9. (9a) Numerical solution with the triangular fuzzy number with $K_1 = 74692359.51$, $K_2 = 75692359.51$ and $K_3 = 76692359.51$ for $T = 10$; (9b) Graph of triangle membership function for the model solution with $T = 10$; (9c) Graph of triangle membership function for the model solution with $T = 0$

(10a) (10b) (10c)

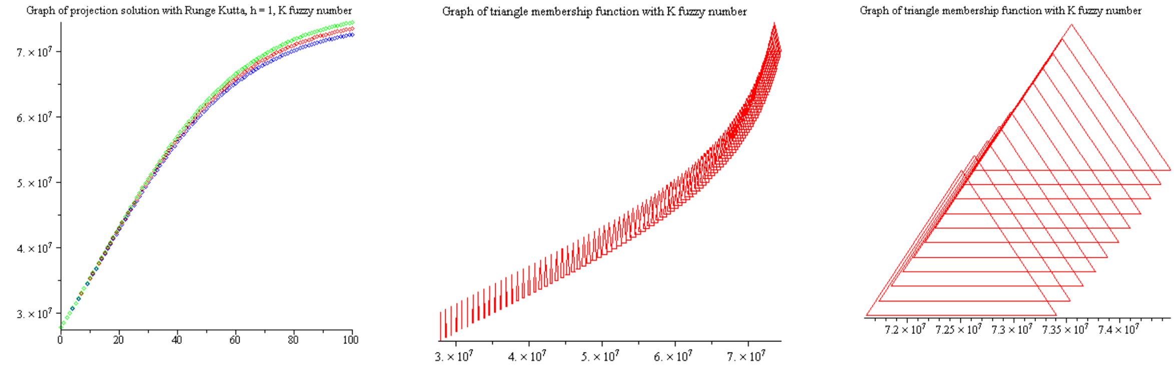


Figure 10. (10a) Numerical solution with the triangular fuzzy number with $K_1 = 74692359.51$, $K_2 = 75692359.51$ and $K_3 = 76692359.51$; (10b) Graph of triangle membership function for the model solution with $T = 100$; (10c) Graph of triangle membership function for the model solution with $T = 90 \cdots 100$

(11a) (11b) (11c)

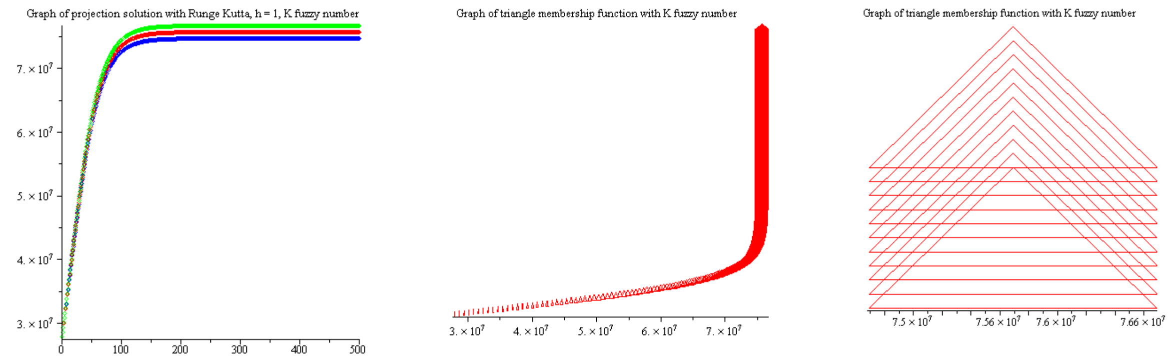


Figure 11. (11a) Numerical solution with the triangular fuzzy number with $K_1 = 74692359.51$, $K_2 = 75692359.51$ and $K_3 = 76692359.51$; (11b) Graph of triangle membership function for the model solution with $T = 500$; (11c) Graph of triangle membership function for the model solution with $T = 490 \cdots 500$

While in Figures (9b), (10b), and (11b) it is illustrated the graph of triangle membership function for the model solution for projection up to 10, 100 and 500 iterations of time. In Figure (9c) it is shown the graph of triangle membership function at iteration 0. It shows that it does not contain uncertainty on data. Figure (10c) is a graph of triangle membership function at iteration 90 to 100 and figure (11c) is a graph of triangle membership function at iteration 490 to 500 where uncertainty becomes apparent.

4. Conclusion

The population growth model used in this paper is the Logistic model. The value of carrying capacity (K), and growth rate (r) is obtained from the best sampling interval projection on the growth model of population logistic in West Java Province. We then look for the numeric solution of the model by using fourth-order Runge-Kutta method. To overcome uncertainty on the data, the fuzzy set theory approach is used. In this case, the initial population number (P_0), growth rate (r) and carrying capacity (K) are approximated by the triangular membership function. From the result of the discussion, it can be concluded that as far as mathematical model concern, fuzzy number initial values are affected the fuzziness of the projection, but in the case of logistic model, this fuzziness is dissapearing in the long-term. We also noted that fuzzy theory approach is considered appropriate to overcome uncertainty on the data in solving numerical solution of the model.

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