Eigenvectors determination of the ribosome dynamics model during mRNA translation using the Kleene Star algorithm

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Abstract. Eigenvalues and eigenvectors in max-plus algebra have the same important role as eigenvalues and eigenvectors in conventional algebra. In max-plus algebra, eigenvalues and eigenvectors are useful for knowing dynamics of the system such as in train system scheduling, scheduling production systems and scheduling learning activities in moving classes. In the translation of proteins in which the ribosome move uni-directionally along the mRNA strand to recruit the amino acids that make up the protein, eigenvalues and eigenvectors are used to calculate protein production rates and density of ribosomes on the mRNA. Based on this, it is important to examine the eigenvalues and eigenvectors in the process of protein translation. In this paper an eigenvector formula is given for a ribosome dynamics during mRNA translation by using the Kleene star algorithm in which the resulting eigenvector formula is simpler and easier to apply to the system than that introduced elsewhere. This paper also discusses the properties of the matrix $B_\lambda^{\otimes n}$ of model. Among the important properties, it always has the same elements in the first column for $n = 1, 2, \ldots$ if the eigenvalue is the time of initiation, $\lambda = \tau_{in}$, and the column is the eigenvector of the model corresponding to λ.

1. Introduction

Max-plus algebra is widely used to model time events discrete system. A typical application of discrete events system are production lines, where every machine must wait with a starting a new operation until the operations on other machines are completed as in the scheduling, transporting, manufacturing, queuing, and protein production in cell. One problem that often appears in the max-plus algebra is the eigen problem. The eigen problem for max-plus matrices describes the steady state of the system. A more details examples of the eigen problems can be seen in [4], [5], [7], [9], [10], [11].

The eigenvalues and eigenvectors in a max-plus model can be used to get an overview of the system dynamics such as in train system scheduling, scheduling production systems, and scheduling lesson activities on moving classes. In the translation stage of cellular protein production, in which the ribosomes moves uni-directionally along an mRNA strand to building amino acid chains, the eigenvalues and eigenvectors are useful to calculate the protein production rate and density of ribosomes on the mRNA. Based on this, it is important to examine the eigenvalues and eigenvectors of the model.

In the previous work, the authors in [3] provided a formula for determining the eigenvalues and eigenvectors of their model [3]. In this paper we will discuss a simpler formula that is easier to understand the process of determining the eigenvectors of their model. This formula is obtained by a different approach compared to the work in [3].

2. Model and Method
The details derivation of the model are given in [3]. In this section we only present a brief theoretical background required in the subsequent sections, such as mRNA translation, max-plus algebra, eigen problem in max-plus algebra and matrices of the ribosome dynamics during mRNA translation.

2.1. Messenger RNA translation
mRNA translation is one of the steps in protein production in cells [1]. The mRNAs are single strands of nucleotides, grouped in triplets called codons, holds the code for a specific chain of amino acids that makes up a protein. The translation is performed by molecular machines called ribosomes, which scan along the mRNA adding amino acids to a growing chain which will become the protein. In this paper we will examine the eigenvalues and eigenvectors of the matrix B in (1) which is a max-plus model of ribosome dynamics during mRNA translation introduced by [3]. See [3] for the details.

$$B = \begin{bmatrix} \tau_{in} & 0 & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \tau_{in} \otimes \tau_1 & \tau_1 & 0 & \varepsilon & \cdots & \varepsilon \\ \tau_{in} \otimes \tau_1 \otimes \tau_2 & \tau_1 \otimes \tau_2 & \tau_2 & 0 & \ddots & \vdots \\ \tau_{in} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 & \tau_1 \otimes \tau_2 \otimes \tau_3 & \tau_2 \otimes \tau_3 & \ddots & \ddots & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \tau_{n-1} & 0 \\ \tau_{in} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \cdots & \tau_{n-1} \otimes \tau_n & \tau_n \oplus \tau_{out} \end{bmatrix} \tag{1}$$

2.2. Max-plus algebra
Bacelli *et al.* in [2] defines the max-plus algebra and denoted by $\mathcal{R}_{max}$ which is defined as the set $\mathbb{R}_{max} = \mathbb{R} \cup -\infty$, with $\mathbb{R}$ is real numbers extended by an infinite element $\varepsilon = -\infty$ and operation $\oplus$, $\otimes$ are defined by

$$a \oplus b = \max\{a,b\} \text{ and } a \otimes b = a+b \tag{2}$$

where $a, b \in \mathbb{R}_{\max}$. It is easy to show that both operation $\oplus$ and $\otimes$ are associative and commutative. The zero and unit element in max-plus algebra are $\varepsilon = -\infty$ and $e = 0$, respectively.
We define max-plus power in the natural way

$$a^{\otimes k} = a \otimes a \otimes a \otimes a \otimes \ldots \otimes a \text{ , } k \text{ times} \tag{3}$$

We can write $a^{\otimes k}$ in conventional algebra as $k \times a$. We also note that the $\otimes$ operator has an inverse which can be expressed as a negative power

$$a \otimes b^{\otimes (-1)} = a - b, \tag{4}$$

but there is no inverse of the $\oplus$ operator.

Vectors and matrices can also be constructed. We denoted by a_{ij} the i-jth component of the matrix A. For matrices $A \in \mathbb{R}_{max}^{n \times m}, B \in \mathbb{R}_{max}^{n \times m}, C \in \mathbb{R}_{max}^{m \times l}$ the sum and product of matrices are then defined as

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij}, \tag{5}$$

$$[A \otimes C]_{ij} = \bigoplus_{k=1}^{m} a_{ik} \otimes c_{kj}, \tag{6}$$

where $a_{ij} = [A]_{ij}$, $b_{ij} = [B]_{ij}$ and $c_{ij} = [C]_{ij}$, and have used the following convenient notation for summation over indices:

$$\bigoplus_{i=1}^{n} a_i = \max\{a_i | 1 \le i \le n\}.$$

For square matrix $A \in \mathbb{R}_{max}^{n \times n}$, similar to scalar in max-plus algebra, we denote

$A^{\otimes k} = \underbrace{A \otimes A \otimes ... \otimes A}_{k} = A \otimes A^{k-1}$ as k^{th} power of A, while $A^{+} = A \oplus A^2 \oplus A^3 \oplus ... = \oplus_{k=1}^{\infty} A^k$ and

$A^{*} = E_n \oplus A^{+}$ with E_n is the identity matrix in $\mathbb{R}_{max}^{n \times n}$. Furthermore, those relation can be written in the form of $A^{*} = \oplus_{k=0}^{\infty} A^{\otimes k}$ which is called *Kleene star*. Therefore,

$$A^{+} = A \otimes A^{*} \tag{7}$$

Eigen problems are common problem in mathematics especially in linear algebra. In max-plus algebra, similar to linear algebra, eigen problems are formulated as $A \otimes \mathbf{u} = \lambda \otimes \mathbf{u}$ for given matrix $A \in \mathbb{R}_{max}^{n \times n}$, where λ is called *eigenvalue* and $\mathbf{u} \in \mathbb{R}_{max}^{n}$ is called *eigenvector*. This problem is treated by several authors who can be such as in [4], [5], [10], [11]. The graph associated with matrix $A \in \mathbb{R}_{max}^{n \times n}$, is denoted by $G(A)$. It consists of a set of nodes and directed weighted arcs, where if there is a matrix element $[A]_{ij}$ with value $a_{ij} \neq \varepsilon$, then there is an arc $j \rightarrow i$ (from j to i) with weight a_{ij}. A series of one or more arcs between two nodes i and j is called a *path* $i \rightarrow j$. A path $i \rightarrow i$ is called a *circuit* and denoted γ. A graph is said to be *strongly connected* if for any two different nodes there is a path between them and a matrix A is said to be *irreducible* if the corresponding graph $G(A)$ is strongly connected. The *circuit weight* w_{γ} of a circuit γ is defined as the sum of the weight of all arcs in that circuit and the *circuit length* l_{γ} as the number of arcs in the circuit. The *mean circuit weight* is then defined as $\overline{w}_{\gamma} = \dfrac{w_{\gamma}}{l_{\gamma}}$. If the maximum mean circuit weight in a graph is λ, then a circuit with a mean circuit weight equal to λ is called a *critical circuit*. The *critical graph* corresponding to matrix A is denoted by $G^{c}(A)$. It is defined as the subgraph of $G(A)$ containing only the nodes and arcs which are in the critical circuits and the set of nodes in $G(A)$ denoted by $U^{c}(A)$. Furthermore, in this paper we define $\left(A_{\lambda}\right)_{ij} \in \mathbb{R}_{max}^{n \times n}$ as

$$\left(A_{\lambda}\right)_{ij} = \left(-\lambda \otimes A\right)_{ij} \tag{8}$$

for any λ. For the analysis of the subsequent section we need the following theorems:

Theorem 1. [6] *Let $A \in \mathbb{R}_{max}^{n \times n}$ be a matrix that has a maximal mean circuit weight of circuits in $G(A)$ equal to e then A^{+} exists and is defined by*

$$A^{+} = A \oplus A^2 \oplus ... \oplus A^n = \oplus_{k=1}^{n} A^{\otimes k} .$$

Based on Theorem 1 it is given that $A^{+} = A \oplus A^2 \oplus ... \oplus A^n = A \otimes \left(E_n \oplus A \oplus A^2 \oplus ... \oplus A^{n-1}\right)$, and according to (7) it is obtained that

$$A^{*} = E_n \oplus A \oplus A^2 \oplus ... \oplus A^{n-1} . \tag{9}$$

Theorem 2. *Let $A \in \mathbb{R}_{max}^{n \times n}$ has finite number of the maximal mean circuit weight λ, then λ is the eigenvalue of A and for any $\mathbf{u} \in U^{c}$, the column of $\left(A_{\lambda}^{*}\right)_{\mathbf{u}}$ is the eigenvector of A corresponding to the λ.*
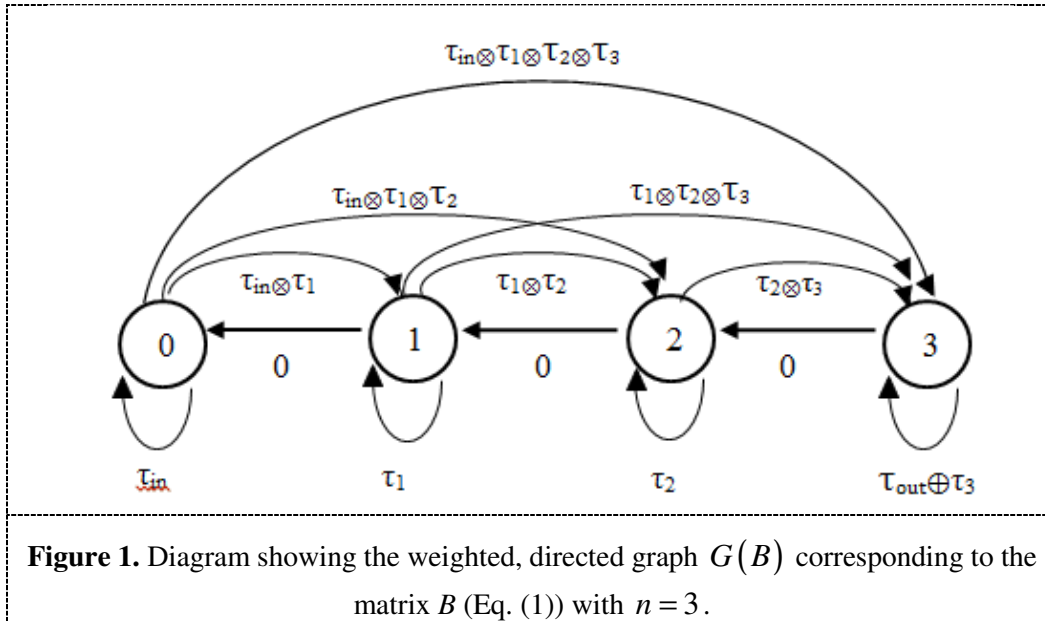
Theorem 3. [8] *Any irreducible matrix $A \in \mathbb{R}_{max}^{n \times n}$ posseses one and only one eigenvalue λ, which is a finite number equal to the maximal mean circuit weight of circuits in the graph $G(A)$.*

Theorem 4. *Let $B \in \mathbb{R}_{max}^{(n+1) \times (n+1)}$ is given by (1), then $\lambda = \max \left\{ \tau_{in}, \tau_{out}, \tau_i \,\middle|\, 1 \leq i \leq n \right\}$.*

We do not prove the theorems. Interested readers can consult the respective references.

3. Result and Discussion

In this section, we will examine how to determine the eigenvector of the matrix B in (1). The eigenvalue of the matrix B can be found from the correspomding graph $G(B)$. Fig. 1 show the example of graph $G(B)$ with $n = 3$.

Figure 1. Diagram showing the weighted, directed graph $G(B)$ corresponding to the matrix B (Eq. (1)) with $n = 3$.

The matrix B is irreducible and has exactly one eigenvalue because the graph $G(B)$ is strongly connected. Since the eigenvalue of B is the maximum mean circuit weight, then the eigenvalue of B denotes the largest waiting time of the model. In [3] the solution of determining the eigenvector depends on the corresponding eigenvalues of Theorem 4 as follows:

a. If $\lambda = \tau_{in}$ then the final expression for the eigenvector is

$$\mathbf{u} = \begin{bmatrix} \tau_1^{\otimes(-1)} \otimes \cdots \otimes \tau_n^{\otimes(-1)} \otimes u_n \\ \tau_2^{\otimes(-1)} \otimes \cdots \otimes \tau_n^{\otimes(-1)} \otimes u_n \\ \vdots \\ \tau_n^{\otimes(-1)} \otimes u_n \\ u_n \end{bmatrix}. \tag{10}$$

b. If $\lambda = \tau_{out}$ then the final expression for the eigenvector is

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \tau_{out} \otimes u_0 \\ \tau_{out}^{\otimes 2} \otimes u_0 \\ \vdots \\ \tau_{out}^{\otimes n} \otimes u_0 \end{bmatrix}. \tag{11}$$

c. If $\lambda = \max\{\tau_i \mid 1 \leq i \leq n\}$ then the final expression for the eigenvector is

$$\mathbf{u} = \begin{bmatrix} u_0 \\ \tau \otimes u_0 \\ \tau^{\otimes 2} \otimes u_0 \\ \vdots \\ \tau^{\otimes q} \otimes u_0 \\ \tau_{q+1} \otimes \tau^{\otimes q} \otimes u_0 \\ \tau_{q+1} \otimes \tau_{q+2} \otimes \tau^{\otimes q} \otimes u_0 \\ \vdots \\ \tau_{q+1} \otimes \cdots \otimes \tau_n \otimes \tau^{\otimes q} \otimes u_0 \end{bmatrix}, \tag{12}$$

and
$$\mathbf{u} = \begin{bmatrix} u_0 \\ \tau \otimes u_0 \\ \tau^{\otimes 2} \otimes u_0 \\ \vdots \\ \tau^{\otimes p} \otimes u_0 \\ \tau_{p+1} \otimes \tau^{\otimes p} \otimes u_0 \\ \tau_{p+1} \otimes \tau_{p+2} \otimes \tau^{\otimes p} \otimes u_0 \\ \vdots \\ \tau_{p+1} \otimes \cdots \otimes \tau_n \otimes \tau^{\otimes p} \otimes u_0 \end{bmatrix}. \tag{13}$$

The next discussion is purported to calculate the eigenvectors which different to the method in [3]. It is done by using the Kleene star algorithm. We identify the eigenvectors into three cases: $\lambda = \tau_{in}$, $\lambda = \tau_{out}$, $\lambda = \max\{\tau_i \mid 1 \le i \le n\}$. The steps of determining the eigenvector formula of the matrix B in (1) by the Kleene star method is undertaken with the following steps:

    i.   Determine the eigenvalues of B by the formula $\lambda = \max\{\tau_{in}, \tau_{out}, \tau_i \mid 1 \le i \le n\}$

    ii.   Calculate $B_\lambda = -\lambda \otimes B$

    iii.   Calculate $B_\lambda^* = E_n \oplus B_\lambda \oplus B_\lambda^{\otimes 2} \oplus B_\lambda^{\otimes 3} \oplus \ldots \oplus B_\lambda^{\otimes n}$

    iv. Choose the eigenvector of matrix B of the column vectors in B_λ^* corresponding to the location of the eigenvalues in matrix B.

Several other authors have reviewed this method such as [8]. By using this method we find the following theorem regarding the properties of the matrix $B_\lambda^{\otimes n}$ of (1) as follows:

Theorem 5. Let $B_{\max}^{(n+1)\times(n+1)}$ satisfy (1) with $\lambda = \tau_{in}$, then

    i. $B_\lambda^{\otimes n}$ is the matrix with the first column always the same for every n, where $n = 1, 2, 3, \ldots$

    ii. b_{1n} is the first column of $B_\lambda^{\otimes n}$ and it is the eigenvector of B.

Proof:
    i.   For $n = 1, 2, 3, \ldots$ we obtain

$$B_\lambda = \begin{bmatrix} 0 & -\tau_{in} & \varepsilon & \cdots & \varepsilon \\ \tau_1 & -\tau_{in} \otimes \tau_1 & -\tau_{in} & \cdots & \varepsilon \\ \tau_1 \otimes \tau_2 & -\tau_{in} \otimes \tau_1 \otimes \tau_2 & -\tau_{in} \otimes \tau_2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & -\tau_{in} \\ \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n & -\tau_{in} \otimes \tau_1 \otimes \cdots \otimes \tau_n & -\tau_{in} \otimes \tau_2 \otimes \cdots \otimes \tau_n & \cdots & -\tau_{in} \otimes (\tau_n \oplus \tau_{out}) \end{bmatrix}$$

$$B_\lambda^{\otimes 2} = \begin{bmatrix} 0 & -\tau_{in} & -2\tau_{in} & \varepsilon & \cdots & \varepsilon \\ \tau_1 & -\tau_{in} \otimes \tau_1 & -2\tau_{in} \otimes (\tau_1 \oplus \tau_2) & -2\tau_{in} & \cdots & \varepsilon \\ \tau_1 \otimes \tau_2 & -\tau_{in} \otimes \tau_1 \otimes \tau_2 & \cdots & -2\tau_{in} \otimes (\tau_2 \oplus \tau_3) & \ddots & \vdots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 & -\tau_{in} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 & \cdots & -2\tau_{in} \otimes \tau_3 \otimes (\tau_2 \oplus \tau_3) & \ddots & \varepsilon \\ \vdots & \vdots & \vdots & \vdots & \vdots & -2\tau_{in} \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & -\tau_{in} \otimes \tau_1 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$B_\lambda^{\otimes 3} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 \otimes \tau_2 & \cdots & \cdots & \cdots & \ddots & \vdots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 & \cdots & \cdots & \cdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} ; \quad B_\lambda^{\otimes 4} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 \otimes \tau_2 & \cdots & \cdots & \cdots & \ddots & \vdots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 & \cdots & \cdots & \cdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$B_\lambda^{\otimes 5} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 \otimes \tau_2 & \cdots & \cdots & \cdots & \ddots & \vdots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 & \cdots & \cdots & \cdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} ; \quad B_\lambda^{\otimes 6} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 & \cdots & \cdots & \cdots & \cdots & \varepsilon \\ \tau_1 \otimes \tau_2 & \cdots & \cdots & \cdots & \ddots & \vdots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 & \cdots & \cdots & \cdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

and so on. It can be seen that all the first columns of the matrix $B_\lambda^{\otimes n}$ have elements that are always the same for $n = 1,2,3,...$ ■

ii. $B_\lambda^* = E_n \oplus B_\lambda \oplus B_\lambda^{\otimes 2} \oplus \cdots$

$$B_\lambda^* = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \cdots & \varepsilon \\ \varepsilon & 0 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \cdots & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots \\ \tau_1 & \cdots & \cdots & \cdots & \cdots \\ \tau_1 \otimes \tau_2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_1 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \oplus \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots \\ \tau_1 & \cdots & \cdots & \cdots & \cdots \\ \tau_1 \otimes \tau_2 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_1 \otimes \cdots \otimes \tau_n & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \oplus \cdots$$

Since $\lambda = \tau_{in}$ in column 1 of matrix B corresponds to Theorem 2, the eigenvector is the first column of the matrix B_λ^* as follows

$$b_{1n} = \begin{bmatrix} 0 \\ \tau_1 \\ \tau_1 \otimes \tau_2 \\ \vdots \\ \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n \end{bmatrix}$$
                                                                                                      ■

In the next section will examine the eigenvectors of the matrix (1) by the Kleene star algorithm .

3.1. Case (i): $\lambda = \tau_{in}$

In this case we show that the waiting time for initiation of translations is longer that for tRNA capture and termination, i.e. $\tau_{in} > \tau_{out}, \tau_i$ for $1 \le i \le n$.

   a. $\lambda = \tau_{in}$

   b. By using Theorem 5 the resulting eigenvector of matrix (1) is

$$\mathbf{u} = \begin{bmatrix} 0 \\ \tau_1 \\ \tau_1 \otimes \tau_2 \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \\ \vdots \\ \tau_1 \otimes \tau_2 \otimes \tau_3 \otimes ... \otimes \tau_n \end{bmatrix} \tag{14}$$

3.2. Case (ii): $\lambda = \tau_{out}$

In this case we show that the waiting for termination of translation is longer that for tRNA capture and initiation, i.e. $\tau_{out} > \tau_{in}, \tau_i$ for $1 \le i \le n$.

   a. $\lambda = \tau_{out}$

   b. $B_\lambda = \begin{bmatrix} -\tau_{out} \otimes \tau_{in} & -\tau_{out} & \varepsilon & \varepsilon \\ -\tau_{out} \otimes \tau_{in} \otimes \tau_1 & -\tau_{out} \otimes \tau_1 & -\tau_{out} & \varepsilon \\ -\tau_{out} \otimes \tau_{in} \otimes \tau_1 \otimes \tau_2 & -\tau_{out} \otimes \tau_1 \otimes \tau_2 & -\tau_{out} \otimes \tau_2 & -\tau_{out} \\ -\tau_{out} \otimes \tau_{in} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 & -\tau_{out} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 & -\tau_{out} \otimes \tau_2 \otimes \tau_3 & 0 \end{bmatrix}$

$$B_\lambda^{\otimes 2} = \begin{bmatrix} -2\tau_{out} \otimes \tau_{in} \otimes (\tau_{in} \oplus \tau_1) & -2\tau_{out} \otimes (\tau_{in} \oplus \tau_1) & -2\tau_{out} & \varepsilon \\ -2\tau_{out} \otimes \tau_{in} \otimes \tau_1 \otimes (\tau_{in} \oplus \tau_1 \oplus \tau_2) & -2\tau_{out} \otimes \tau_1 \otimes (\tau_{in} \oplus \tau_1 \oplus \tau_2) & -2\tau_{out} \otimes (\tau_1 \oplus \tau_2) & -2\tau_{out} \\ -2\tau_{out} \otimes \tau_{in} \otimes \tau_1 \otimes \tau_2 (\tau_{in} \oplus \tau_1 \oplus \tau_2 \oplus \tau_3) & -2\tau_{out} \otimes \tau_1 \otimes \tau_2 (\tau_{in} \oplus \tau_1 \oplus \tau_2 \oplus \tau_3) & -2\tau_{out} \otimes \tau_2 \otimes (\tau_1 \oplus \tau_2 \oplus \tau_3) & -\tau_{out} \\ -\tau_{out} \otimes \tau_{in} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 & -\tau_{out} \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 & -\tau_{out} \otimes \tau_2 \otimes \tau_3 & 0 \end{bmatrix}$$

$$B_\lambda^{\otimes 3} = \begin{bmatrix} \cdots & \cdots & \cdots & -3\tau_{out} \\ \cdots & \cdots & \cdots & -2\tau_{out} \\ \cdots & \cdots & \cdots & -\tau_{out} \\ \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

   c. $B_\lambda^* = E_n \oplus B_\lambda \oplus B_\lambda^{\otimes 2} \oplus B_\lambda^{\otimes 3}$

Since $\lambda = \tau_{out}$ in column 4 of matrix B corresponds to theorem 2, the eigenvector is the fourth column of the matrix B_λ^*. The resulting eigenvector is

$$\mathbf{u} = \begin{bmatrix} -3\tau_{out} \\ -2\tau_{out} \\ -\tau_{out} \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_{out}^{\otimes(-3)} \\ \tau_{out}^{\otimes(-(3-1))} \\ \tau_{out}^{\otimes(-(3-2))} \\ 0 \end{bmatrix}.$$

By continuing this step for the matrix $B_{\max}^{(n+1) \times (n+1)}$ we obtain a general formula for eigenvector associated with $\lambda = \tau_{out}$ as follows

$$\mathbf{u} = \begin{bmatrix} \tau_{out}^{\otimes(-n)} \\ \tau_{out}^{\otimes(-(n-1))} \\ \tau_{out}^{\otimes(-(n-2))} \\ \tau_{out}^{\otimes(-(n-3))} \\ \vdots \\ 0 \end{bmatrix}. \tag{15}$$

3.3. Case (iii): $\lambda = \max\{\tau_i \,|\, 1 \le i \le n\}$

In this case we assume that there are two places in the bulk of the mRNA, with waiting times equals to τ. That is, there exist integers p and q such that $\tau_p, \tau_q = \tau$, where $1 < p < q < n$.

a. $\lambda = \max\{\tau_i \,|\, 1 \le i \le n\}$

b.   For simplicity we use $n = 4$ so

$$B_\lambda = \begin{bmatrix} -\tau \otimes \tau_{in} & -\tau & \varepsilon & \varepsilon & \varepsilon \\ -\tau \otimes \tau_{in} \otimes \tau_1 & -\tau \otimes \tau_1 & -\tau & \varepsilon & \varepsilon \\ \tau_{in} \otimes \tau_1 & \tau_1 & 0 & -\tau & \varepsilon \\ \tau_{in} \otimes \tau_1 \otimes \tau & \tau_1 \otimes \tau & \tau & 0 & -\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau \otimes \tau_4 & \tau_1 \otimes \tau \otimes \tau_4 & \tau \otimes \tau_4 & \tau_4 & -\tau \otimes (\tau_4 \oplus \tau_{out}) \end{bmatrix}$$

$$B_\lambda^{\otimes 2} = \begin{bmatrix} -2\tau \otimes \tau_{in} \otimes (\tau_{in} \oplus \tau_1) & -2\tau \otimes (\tau_{in} \oplus \tau_1) & -2\tau & \varepsilon & \varepsilon \\ -\tau \otimes \tau_{in} \otimes \tau_1 & -\tau \otimes \tau_1 & -\tau & -2\tau & \varepsilon \\ \tau_{in} \otimes \tau_1 & \tau_1 & 0 & -\tau & -2\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau & \tau_1 \otimes \tau & \tau & 0 & -\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau \otimes \tau_4 & \tau_1 \otimes \tau \otimes \tau_4 & \tau \otimes \tau_4 & \tau_4 & -\tau \otimes \tau_4 \end{bmatrix}$$

$$B_\lambda^{\otimes 3} = \begin{bmatrix} -2\tau \otimes \tau_{in} \otimes \tau_1 & -2\tau \otimes \tau_1 & -2\tau & -3\tau & \varepsilon \\ -\tau \otimes \tau_{in} \otimes \tau_1 & -\tau \otimes \tau_1 & -\tau & -2\tau & -3\tau \\ \tau_{in} \otimes \tau_1 & \tau_1 & 0 & -\tau & -2\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau & \tau_1 \otimes \tau & \tau & 0 & -\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau \otimes \tau_4 & \tau_1 \otimes \tau \otimes \tau_4 & \tau \otimes \tau_4 & \tau_4 & -\tau \otimes \tau_4 \end{bmatrix}$$

$$B_\lambda^{\otimes 4} = \begin{bmatrix} -2\tau \otimes \tau_{in} \otimes \tau_1 & -2\tau \otimes \tau_1 & -2\tau & -3\tau & -4\tau \\ -\tau \otimes \tau_{in} \otimes \tau_1 & -\tau \otimes \tau_1 & -\tau & -2\tau & -3\tau \\ \tau_{in} \otimes \tau_1 & \tau_1 & 0 & -\tau & -2\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau & \tau_1 \otimes \tau & \tau & 0 & -\tau \\ \tau_{in} \otimes \tau_1 \otimes \tau \otimes \tau_4 & \tau_1 \otimes \tau \otimes \tau_4 & \tau \otimes \tau_4 & \tau_4 & -\tau \otimes \tau_4 \end{bmatrix}.$$

c. $B_\lambda^* = E_n \oplus B_\lambda \oplus B_\lambda^{\otimes 2} \oplus B_\lambda^{\otimes 3} \oplus B_\lambda^{\otimes 4}$

Now, focus on the third and fourth columns of the matrix B_λ^* corresponds to Theorem 2 about the existence of its eigenvalues in matrix B. The resulting eigenvector is

$$\mathbf{u}_{\tau_p} = \begin{bmatrix} -2\tau \\ -\tau \\ 0 \\ \tau \\ \tau_4 \otimes \tau \end{bmatrix} = \begin{bmatrix} \tau^{\otimes 0} \otimes \tau^{\otimes(-2)} \\ \tau^{\otimes 1} \otimes \tau^{\otimes(-2)} \\ \tau^{\otimes 2} \otimes \tau^{\otimes(-2)} \\ \tau_{2+1} \otimes \tau^{\otimes 2} \otimes \tau^{\otimes(-2)} \\ \tau_{2+2} \otimes \tau_{2+1} \otimes \tau^{\otimes 2} \otimes \tau^{\otimes(-2)} \end{bmatrix} = \begin{bmatrix} \tau^{\otimes 0} \otimes \tau^{\otimes(-p)} \\ \tau^{\otimes 1} \otimes \tau^{\otimes(-p)} \\ \tau^{\otimes p} \otimes \tau^{\otimes(-p)} \\ \tau_{p+1} \otimes \tau^{\otimes p} \otimes \tau^{\otimes(-p)} \\ \tau_{p+2} \otimes \tau_{p+1} \otimes \tau^{\otimes p} \otimes \tau^{\otimes(-p)} \end{bmatrix}$$

$$\text{and } \mathbf{u}_{\tau_q} = \begin{bmatrix} -3\tau \\ -2\tau \\ -\tau \\ 0 \\ \tau_4 \end{bmatrix} = \begin{bmatrix} \tau^{\otimes 0} \otimes \tau^{\otimes(-3)} \\ \tau^{\otimes 1} \otimes \tau^{\otimes(-3)} \\ \tau^{\otimes 2} \otimes \tau^{\otimes(-3)} \\ \tau^{\otimes 3} \otimes \tau^{\otimes(-3)} \\ \tau_{3+1} \otimes \tau^{\otimes 3} \otimes \tau^{\otimes(-3)} \end{bmatrix} = \begin{bmatrix} \tau^{\otimes 0} \otimes \tau^{\otimes(-q)} \\ \tau^{\otimes 1} \otimes \tau^{\otimes(-q)} \\ \tau^{\otimes 2} \otimes \tau^{\otimes(-q)} \\ \tau^{\otimes q} \otimes \tau^{\otimes(-q)} \\ \tau_{q+1} \otimes \tau^{\otimes q} \otimes \tau^{\otimes(-q)} \end{bmatrix}.$$

By continuing this step for the matrix $B_{\max}^{(n+1)\times(n+1)}$ we obtain a general formula for eigenvector associated with $\lambda = \max\{\tau_i \mid 1 \le i \le n\}$ as follows

$$\mathbf{u}_{\tau_p} = \begin{bmatrix} \tau^{\otimes 0} \otimes \tau^{\otimes(-p)} \\ \tau^{\otimes 1} \otimes \tau^{\otimes(-p)} \\ \vdots \\ \tau^{\otimes p} \otimes \tau^{\otimes(-p)} \\ \tau_{p+1} \otimes \tau^{\otimes p} \otimes \tau^{\otimes(-p)} \\ \vdots \\ \tau_{p+1} \otimes \dots \otimes \tau_n \otimes \tau^{\otimes p} \otimes \tau^{\otimes(-p)} \end{bmatrix}, \tag{16}$$

and

$$\mathbf{u}_{\tau_q} = \begin{bmatrix} \tau^{\otimes 0} \otimes \tau^{\otimes(-q)} \\ \tau^{\otimes 1} \otimes \tau^{\otimes(-q)} \\ \vdots \\ \tau^{\otimes q} \otimes \tau^{\otimes(-q)} \\ \tau_{q+1} \otimes \tau^{\otimes q} \otimes \tau^{\otimes(-q)} \\ \vdots \\ \tau_{q+1} \otimes \dots \otimes \tau_n \otimes \tau^{\otimes q} \otimes \tau^{\otimes(-q)} \end{bmatrix}. \tag{17}$$

4. Conclusion

This paper shows the formula for determining the eigenvectors of the system introduced in [3] using the Kleene star algorithm, where the eigenvector formula produced in this study is simpler and easier to use in finding eigenvectors of the matrix B. In this paper we also show the properties of the matrix $B_\lambda^{\otimes n}$ that if the eigenvalues of the matrix B is $\lambda = \tau_{in}$ then the first column of the matrix $B_\lambda^{\otimes n}$ is always the same for every $n = 1, 2, \dots$ and they becomes the eigenvector of the matrix B.

Acknowledgment

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