Nonlinear Diophantine equation $11^x + 13^y = z^2$

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Abstract. This research aims to obtaining the solutions (if any) from the Non Linear Diophantine equation of $11^x + 13^y = z^2$. There are 3 possibilities to obtain the solutions (if any) from the Non Linear Diophantine equation, namely single, multiple, and no solution. This research is conducted in two stages: (1) by utilizing simulation to obtain the solutions (if any) from the Non Linear Diophantine equation of $11^x + 13^y = z^2$ and (2) by utilizing congruency theory with its characteristics proven that the Non Linear Diophantine equation has no solution for non negative whole numbers (integers) of x, y, z .

Keywords: non linear Diophantine equation, simulation, congruency theory

1. Introduction

The congruency theory was first found by Carl Friedrich Gauss (1977-1855) which is considered as one monumental findingand becomes the foundation of the modern number theory. Congruency is actually developed from the divisibility relation then further developed by Diophantine called the linear congruency theory. The linear congruency theory focuses on finding the solution (if any) from the equation in the form of $ax \equiv b\text{mod}(n)$. It means that if x_0 is the solution of the linear congruency $ax \equiv b\text{mod}(n)$ then $ax_0 \equiv b \text{ mod}(n)$ is fulfilled. Fermat (1980) introduces the nonlinear Diophantine equation in the form $x^n + y^n = z^n$. Especially forn = 2, then all solutions are called the triple Pythagoras.

The forms of nonlinear Diophantine equation continuously develop in line with the involvement of researches on the nonlinear Diophantine equation. Acu [1] proves that (3,0,3) and (2,1,3) are the solutions of the nonlinear Diophantine equation $2^x + 5^y = z^2$.

Sroysang [3] finds that a single solution of the nonlinear Diophantine equation $3^x + 5^y = z^2$ is (1,0,2). Sroysang [7] then finds three solutions of the nonlinear Diophantine equation $2^x + 3^y = z^2$, that is, (0,1,2), (3,0,3), and (4,2,5). Tanakan [6] finds a single solution of the nonlinear Diophantine equation $19^x + 2^y = z^2$, that is, (0,3,3). The above research results lead to a question in order to find the other solution forms of the nonlinear Diophantine equation $11^x + 13^y = z^2$ where *x, y, z* are the non negative whole numbers.

2. Result and discussion

Theorem 1. Assuming that p is the prime number, then the Diophantine equation $p^y+1 = z^2$ has exactly 2 solutions of non negative whole numbers (*p,y,z*). Both solutions are (2,3,3) and (3,1,2).

Proof: for the case $y = 0$ and $z = 0$, there is a contradiction. For the case *y, z*> 0 then it may be written with $z^2 - 1 = (z - 1)(z + 1) = p^y$. Furthermore, it is obtained $2 = (z +1) - (z -1) = p^\beta - p^\alpha$ where $\alpha + \beta = y$ and $\alpha < \beta$. There are two possibilities may occur. If $p^\alpha = 1$ and $p^{\beta-\alpha} - 1 = 2$ then consequently$\alpha = 0$ and $0 + \beta = y$, that $\beta = y$. Furthermore, $p^{\beta-0} - 1 = p^\beta - 1 = p^y - 1 = 2$ that p^y=3 =3^1. Thus, it is obtained the solution (*p, y, z*) = (3, 2, 1). On the other side, if $p^\alpha = 2$ and $p^{\beta-\alpha} - 1 = 1$, then $p^\alpha = 2^1$ that it is obtained $p = 2$ and $\alpha = 1$. Thus, itis consequently$p^{\beta-\alpha} - 1 = 1$ that it is obtained$\beta = 2$. Because$\alpha + \beta = y$ that it is obtained $y = 3$. Thus, it is obtained the solution (*p, y, z*) = (2,3,3) and theorem 1 is proven.

Theorem 2. The Diophantine equation $p^y + 1 = z^2$ has no solution of positive whole numbers for the prime numbers *p*> 3.

Proof: based on theorem 1, it is obtained that at the Diophantine equation $p^y + 1 = z^2$, there are two solutions of non negative whole numbers: (2,3,3) and (3,1,2). Thus, it can be concluded that there is no other solution for each prime number p > 3.

Corollary 3. The Diophantine equation $1 + 13^y = z^2$ has no solution of non negative whole numbers.

Proof: Because 13 is the prime number *p*> 3, that according to theorem 2 then there is no solution of non negative whole number which fulfills the Diophantine equation $p^y + 1 = z^2$.

Corollary 4. The Diophantine equation $11^x + 1 = z^2$ has no solution of the non negative whole numbers.

Proof: From the Diophantine equation $p^y +1 = z^2$ if $y = x$ and $p = 11$ then it is obtained the Diophantine equation $11^x + 1 = z^2$ which has no solution of the non negative whole numbers, based on theorem 2, there is no fulfilling solution of the non negative whole numbers.

Case 1. For the case y = 1 then it is obtained that the Diophantine equation is $11^x + 13 = z^2$. By using congruency $13 \equiv 1 \pmod 4$. Furthermore, if *x* has the odd value, it is obtained $11^x \equiv 3 \pmod 4$ and if *x* has the even value, it is obtained $11^x \equiv 1 \pmod 4$. In addition, when *z* has the odd value, it is obtained $z^2 \equiv 1 \pmod 4$, and if *z* has the even value, it is obtained $z^2 \equiv 0 \pmod 4$. Because $13 \equiv 1 \pmod 4$, it is selected *x* with the odd value and z with the even value in such a way that 11^x +

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$13 = z^2 \Leftrightarrow 3(\text{mod}4) + 1(\text{mod}4) \equiv 0(\text{mod}4)$. Assuming that $x = 2k + 1$, k as the non negative whole number and assuming that $z = 2m$, with m as the non negative whole number, it is further obtained that

$$11^x + 13 = z^2$$
$$11^{2k+1} + 13 = (2m)^2$$
$$11^{2k+1} + 13 = 4m^2$$
$$m^2 = \frac{11^{2k+1} + 13}{4}$$
$$m = \sqrt{\frac{11^{2k+1} + 13}{4}}$$

Because

$m = \sqrt{\frac{11^{2k+1} + 13}{4}}$, with k as the non negative whole number. For k as the non negative whole number, it is obtained m as the non negative whole number that contradiction occurs. Consequently for case 1, the Diophantine equation $11^x + 13 = z^2$ has no solution.

To further generalization, for each y of the positive whole number $13^y \equiv 1$ (mod4).

Case 2. For x with the even value and z with the odd value, If x has the even value, it is obtained $11^x \equiv 1(\text{mod}4)$ andthen if z has the odd value, it is obtained $z^2 \equiv 1(\text{mod}4)$. See that $11^x + 13^y = z^2 \Leftrightarrow 1(\text{mod}4) + 1(\text{mod}4) \equiv 2(\text{mod}4)$. On the other side, it is known that if z has the odd value, then $z^2 \equiv 1(\text{mod}4)$. Thus, there is a contradiction. Consequently for case 2, the Diophantine equation $11^x + 13^y = z^2$ has no solution.

Case 3. For x with the even value and z with the even value, if x has the even value, then it is obtained $11^x \equiv 1(\text{mod}4)$ and if z has the even value, then it is obtained$z^2 \equiv 0(\text{mod}4)$. See that $11^x + 13^y = z^2 \Leftrightarrow 1(\text{mod}4) + 1(\text{mod}4) \equiv 2(\text{mod}4)$. On the other side, it is known that if z has the even value, then $z^2 \equiv 0(\text{mod}4)$. Thus, there is a contradiction. Consequently for case 3, the Diophantine equation $11^x + 13^y = z^2$ has no solution.

Case 4.For x with the odd value and z with the odd value, if x has the odd value, then it is obtained $11^x \equiv 3(\text{mod}4)$ and if z has the odd value, then it is obtained $z^2 \equiv 1(\text{mod}4)$. See that $11^x + 13^y = z^2 \Leftrightarrow 3(\text{mod}4) + 1(\text{mod}4) \equiv 0(\text{mod}4)$. On the other side, it is known that if z has the odd value, then $z^2 \equiv 1(\text{mod}4)$. Thus, there is a contradiction. Consequently for case 4, the Diophantine equation $11^x + 13^y = z^2$ has no solution.

Based on those four explanations, it can be concluded that at the Diophantine equation $11^x + 13^y = z^2$ there is no fulfilling non negative whole numbers (x, y, z). On the other words, the Diophantine equation $11^x + 13^y = z^2$ has no solution.

3. Conclusion

The nonlinear Diophantine equation $11^x + 13^y = z^2$ has no solution for non negative whole numbers of *x, y, z*.

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