

Degree of Approximation by a General C_λ - Summability Method

S Sonker, A Munjal

Department of Mathematics, National Institute of Technology Kurukshetra,
Kurukshetra-136119, Haryana, India
Email: alkamunjal8@gmail.com

Abstract. In the present study, two theorems explaining the degree of approximation of signals belonging to the class $Lip(\alpha, p, w)$ by a more general C_λ - method (Summability method) have been formulated. Improved estimations have been observed in terms of $\lambda(n)$; where $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$ as compared to previous studies presented in terms of n . These estimations of infinite matrices are very much applicable in solid state physics which further motivates for an investigation of perturbations of matrix valued functions.

1. Introduction

Let f be a 2π -periodic function, integrable over the interval $[-\pi, \pi]$. Let the trigonometric Fourier series associated with f be

$$s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx), \quad \forall n \geq 1 \text{ with } s_0(f; x) = \frac{a_0}{2}, \quad (1)$$

denotes the $(n + 1)^{\text{th}}$ partial sums, called trigonometric polynomials of degree (or order) n , of the Fourier series of f . The conjugate series of the Fourier series of f is defined by $\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=0}^{\infty} v_n$ and its n^{th} partial sum is defined as

$$\tilde{s}_n(f; x) = \sum_{k=1}^n (b_k \cos kx - a_k \sin kx), \quad \forall n \geq 1 \text{ and } \tilde{s}_n(f; x) = 0, \quad (2)$$

where,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad k = 0, 1, 2, \dots \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \quad k = 1, 2, 3, \dots \quad (3)$$

1.1. Approximation of Functions.

The problem of approximation of a function is to select a function among a well-defined class that closely matches (approximates) a target function. The need for function approximations arises in many branches of mathematical, physical and engineering science. Approximation theory is the branch of analysis that investigates the operation of certain known signals (functions) approximated by a specific class of functions (for example, polynomials, Fourier series etc.) that often have desirable properties (inexpensive computation, continuity, integral and limit values, etc.). Depending on the structure of the domain and co-domain of the problem, several techniques for approximating the function may be applicable.



A real-valued continuous function defined on a bounded interval on the real line can be well approximated arbitrarily using the sup-norm if the order of the polynomial goes to infinity. Even functions that have a discontinuity can be well approximated arbitrarily, if another norm is used instead of the sup-norm. Taylor series expansion is used for the polynomial approximation of the function and the quality of the approximation depends on the number of terms taken.

Approximations of a real-valued continuous function $f(x)$ by an approximation function $T(x)$ (by Fourier series, orthogonal series) have a finite number of terms. The main aim of this study is to solve the engineering problem (a set of sparse or noisy training data points) by approximation of the continuous function using the summability theory and to find the degree of approximation with minimized error. Based on the developed theorems, some theoretical explanations have been given to estimate the degree of approximation and to minimize the error in data.

The major items in an approximation problem are the type of the approximating function applied and measure of the goodness of an approximation. This is also known as the question of choosing the *form* and *norm*. The choice of the approximating function (form) is more important than the choice of a measure of goodness, that is, a distance function or norm that measures the distance between f and f_a .

1.2. Trigonometric Fourier series approximation with the help of summability method (Cesàro sub-methods).

Let f be 2π -periodic (bounded, integrable) and $f \in L_p[0, 2\pi] = L_p$ for $p \geq 1$. Then,

$$s_n(f; x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \sin kx + b_k \cos kx) \equiv \sum_{k=1}^n u_k(f; x) \quad (4)$$

is the partial sum of the first $(n+1)^{\text{th}}$ terms of the Fourier series of $f \in L_p$ ($p \geq 1$) at a point x , then it is known that

$$s_n(f; x) - f(x) = \frac{2}{\pi} \int_0^\pi \phi_x(t) \frac{\sin(n+1/2)t}{2\sin(t/2)} dt \quad (5)$$

where

$$\phi_x(t) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2f(t) \}.$$

(6)

A measurable 2π -periodic function $w: [0, 2\pi] \rightarrow [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero and $f \in L_w^p([0, 2\pi])$ is the weighted Lebesgue space of all measurable 2π -periodic functions if

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p w(x) dx \right)^{1/p} < \infty. \quad (7)$$

A weight function w belongs to the Muckenhoupt class A_p ($1 < p < \infty$), if

$$\sup \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I [w(x)]^{-1/(p-1)} dx \right)^{p-1} < \infty \quad (8)$$

where the supremum is over all intervals I with length $|I| \leq 2\pi$.

Let $f \in L_w^p$ and $w \in A_p$ ($1 < p < \infty$). Then modulus of continuity of the function f is defined by

$$\Omega(f, \delta)_{p,w} = \sup_{|h| \leq \delta} |\Delta_h(f)|_{p,w}, \quad \delta > 0, \quad (9)$$

where

$$\Delta_h(f(x)) = \frac{1}{h} \int_0^h |f(x+t) - f(x)| dt. \quad (10)$$

For $0 < \alpha \leq 1$, the Lipschitz class $Lip(\alpha, p, w)$ is give as,

$$Lip(\alpha, p, w) = \{ f \in L_w^p : \Omega(f, \delta)_{p,w} = O(\delta^\alpha), \delta > 0 \}. \quad (11)$$

Let $T \equiv (a_{n,k})$ be a lower triangular matrix with non-negative entries. For each n , the sequence $\{a_{n,k}\}$ is either non-increasing or non-decreasing in k , $0 \leq k \leq n$, the matrix T is said to have monotonic rows.

A positive sequence $c = \{c_n\}$ is called almost monotonically increasing (or decreasing) if there exists a constant $K=K(c)$, depending on the sequence c only, such that for all $n \geq m$,

$$c_n \geq K c_m, \quad (\text{or } c_n \leq K c_m).$$

Such sequences will be denoted by $c \in \text{AMDS}$ (or $c \in \text{AMIS}$).

If $e \in \text{AMIS}$ (or $e \in \text{AMDS}$), then the sequence e is almost monotonically increasing (or decreasing) mean sequence, denoted by $e \in \text{AMIMS}$ (or $e \in \text{AMDMS}$) and have the following relations,

$$\text{AMIS} \subset \text{AMIMS}$$

and

$$\text{AMDS} \subset \text{AMDMS}.$$

Let E be an infinite subset of N and consider E as strictly increasing sequence of positive integers, say $E = \{\lambda(n)\}_{n=1}^{\infty}$. The Summability method C_{λ} is defined as

$$(C_{\lambda}(x))_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad (12)$$

where $x = (x_k)$ is a sequence of real or complex numbers. It is clear that C_{λ} is regular for any λ . Thus, the C_{λ} -method yields a subsequence of the Cesàro method C_1 . C_{λ} is obtained by deleting a set of rows from Cesàro matrix. The basic properties of C_{λ} -method can be found in [1, 18].

Armitage and Maddox [1] proved inclusion and tauberian results for the C_{λ} - method. Osikiewicz [6] expanded the work of the Armitage and Maddox by examining further inclusion properties of the C_{λ} - method for bounded sequences and its relationship to statistical convergence. Liendler [7] have dropped the monotonicity on generating sequence $\{p_n\}$ and Mittal *et al.* [8] used more general matrix and found some valuable results. In the present study, C_{λ} - method has been used for generalizing the result of Mittal [8].

Several investigators have studied the degree of approximation of a function using different summability methods of series [17, 4]. In a recent work carried out by Nayak *et al.* [9, 10], the rate of convergence of Fourier series in the generalized Holder metric by *DCM* and second-type *DAM* has been studied. A result on the degree of approximation of functions (signals) in Besov space by *Np*-matrix has also been established by Mohanty *et al.* [5].

In 1986, Bor [2] found the relationship between two summability techniques $|C, 1|_k$ and $|\bar{N}, p_n|_k$ and in [3], he generalized a theorem based on minimal set of sufficient conditions by using the $|\bar{N}, p_n|_k$ for infinite series. In 2016, Sonker and Munjal [11] determined a generalized theorem on absolute Cesàro summability with the sufficient conditions for infinite series and in [12], they used the concept of triangle matrices for obtaining the minimal set of sufficient conditions of an infinite series to be bounded. In 2017, Sonker and Munjal [13] found the approximation of the function $f \in Lip(\alpha, p)$ using infinite matrices of Cesàro sub-method and in [14], they obtained boundness conditions of absolute summability $\varphi - |A|_k$ factors. In this way by using the advanced summability method, we can improve the quality of the filters as in [15, 16].

2. Known Results

Mittal *et al.* [8] has generalized and improved the approximation results by dropping monotonicity on the elements of the matrix rows and proved the following theorem.

Theorem 1 [8]: Let $f \in Lip(\alpha, p)$ and let $T = (a_{n,k})$ be an infinite regular triangular matrix.

(i) If $p > 1$, $0 < \alpha < 1$, $\{a_{n,0}\} \in \text{AMS}$ in k and satisfies

$$(n + 1) \max\{a_{n,0}, a_{n,r}\} = O(1), \quad (13)$$

where $r := \left\lceil \frac{n}{2} \right\rceil$, then $\|f - \tau_n(f)\|_p = O(n^{-\alpha})$ is satisfied.

(ii) If $p > 1$, $\alpha = 1$ and $\sum_{k=0}^{n-1} (n-k)|\Delta_k a_{n,k}| = O(1)$, or

(iii) If $p > 1$, $\alpha = 1$ and $\sum_{k=0}^n |\Delta_k a_{n,k}| = O(a_{n,0})$, or

(iv) If $p = 1$, $0 < \alpha < 1$ and $\sum_{k=0}^n |\Delta_k a_{n,k}| = O(a_{n,0})$,

and also $(n+1)a_{n,0} = O(1)$, hold then $\|f - \tau_n(f)\|_p = O(n^{-\alpha})$ is satisfied.

3. Main Results

Many researchers have studied the error estimation $E_n(f)$ through trigonometric-Fourier approximation for the situations in which the summability matrix dropped its monotonicity. In 2017, Sonker and Munjal [13] found the approximation of the function $f \in \text{Lip}(\alpha, p)$ using infinite matrices of Cesàro sub-method. In the present work, some theorems describing the degree of approximation of signals belonging to the class $\text{Lip}(\alpha, p, w)$ have been explained.

Theorem 2: Let $f \in \text{Lip}(\alpha, p, w)$, and $T = (a_{\lambda(n),k})$ be an infinite regular triangular matrix s.t. $a_{n,k} \geq 0$ and $\sum_{k=1}^n a_{n,k} = 1, \forall n, k$. If, for $p > 1, 0 < \alpha < 1$,

(i) $\{a_{\lambda(n),k}\} \in \text{AMDMS}$ in k and

(ii) $(a_{\lambda(n),0}) = O((\lambda(n) + 1)^{-1})$ satisfies,

$$\text{then } \|f - \tau_n^\lambda(f)\|_{p,w} = O((\lambda(n))^{-\alpha}).$$

Theorem 3: Let $f \in \text{Lip}(\alpha, p, w)$, and let $T = (a_{\lambda(n),k})$ be an infinite regular triangular matrix s.t. $a_{n,k} \geq 0$ and $\sum_{k=1}^n a_{n,k} = 1, \forall n, k$. If, for $p > 1, 0 < \alpha < 1$,

(i) $\{a_{\lambda(n),k}\} \in \text{AMIMS}$ in k and

(ii) $(a_{\lambda(n),0}) = O((\lambda(n) + 1)^{-1})$ satisfies,

$$\text{then } \|f - \tau_n^\lambda(f)\|_{p,w} = O((\lambda(n))^{-\alpha}).$$

In this Approximation, the sharpness of the estimation is improved by using the AMIMS (AMDMS) and more general summability method and with the help of trigonometric polynomials.

4. Lemmas

The following lemmas are used in the proof of theorem 3 and 4.

Lemma 1: If $f \in \text{Lip}(\alpha, p)$ for $0 < \alpha \leq 1$ and $p > 1$, then

$$\|f - s_n(f)\|_{p,w} = O(n^{-\alpha}).$$

Lemma 2: Let matrix have AMS rows and satisfy

$$(\lambda(n) + 1)\max\{a_{\lambda(n),0}, a_{\lambda(n),r}\} = O(1).$$

Then, for $0 < \alpha < 1$,

$$\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} (k+1)^{-\alpha} = O((\lambda(n) + 1)^{-\alpha}).$$

Proof: Suppose that the rows of the matrix are AMDS. Then there exists a $K > 0$ such that

$$\begin{aligned}
\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} (k+1)^{-\alpha} &\leq \sum_{k=0}^{\lambda(n)} K a_{\lambda(n),0} (k+1)^{-\alpha} \\
&= K a_{\lambda(n),0} \sum_{k=0}^{\lambda(n)} (k+1)^{-\alpha} \\
&= O(a_{\lambda(n),0} (\lambda(n)+1)^{1-\alpha}) \\
&= O((\lambda(n)+1)^{1-\alpha}).
\end{aligned}$$

A similar argument applies if the rows of the matrix are AMIMS.

5. Proof of Main Theorem

If $p > 1, 0 < \alpha < 1$.

$$\begin{aligned}
\tau_n^\lambda(f) - f &= \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k^\lambda(f) - t_n^\lambda f + (t_n^\lambda - 1)f \\
&= \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} (s_k^\lambda(f) - f) + (t_n^\lambda - 1)f.
\end{aligned}$$

Using Lemmas 1 and 2,

$$\begin{aligned}
\|\tau_n^\lambda(f) - f\|_{p,w} &\leq \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} \|s_k^\lambda(f) - f\|_{p,w} + |t_n^\lambda - 1| \|f\|_{p,w} \\
&= \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} O((k+1)^{-\alpha}) + O((\lambda(n))^{-\alpha}) \\
&= O((\lambda(n)+1)^{-\alpha}) + O((\lambda(n))^{-\alpha}) \\
&= O((\lambda(n))^{-\alpha}).
\end{aligned}$$

Thus, $\|\tau_n^\lambda(f; x) - f(x)\|_{p,w} = O((\lambda(n))^{-\alpha})$.

This completes the proof of theorem.

Remark: If $\lambda(n) = n$, the methods $N_n^\lambda(f; x)$ and $R_n^\lambda(f; x)$ give us classical known Nörlund and Riesz means. For $\tau_n^\lambda(f; x)$, it coincides with Cesàro arithmetic mean.

6. Conclusion

In the present research work, the approximation degree of the function for generalization of the summability methods using the concept of AMIMS (AMDMS) has been established. The work is a motivation for the researchers, interested in studies of function with the help of trigonometric polynomials. By reducing the restrictions of the filter, the functions of the filter (like Automatic compensation, permanently unit power factor, overcome of unbalancing situation, etc.) have been improved. The result presented here has a direct application in electrical engineering, e.g. the electrical load signal can be represented as a summation of sinusoids with different frequency. Using the outcomes from the present research work, the output of the signals can be made stable, bounded and may be used to predict the behavior of the input data and the changes in the complete process.

Acknowledgement

The authors express their sincere gratitude to the Department of Science and Technology (India) for providing financial support to the second author under INSPIRE Scheme (Innovation in Science Pursuit for Inspired Research Scheme).

References

- [1] Armitage D H and Maddox I J 1989 *Analysis* **9** (1-2) 195-204.
- [2] Bor H 1986 *Proc. Amer. Math. Soc.* **98** (1) 81-84.
- [3] Bor H 1986 *Proc. Amer. Math. Soc.* **118** (1) 71-75.
- [4] Nigam H K and Sharma A 2010 *Int. J. Math. & Math.Sci.* **2010** (351016) 7 pages.
- [5] Mohanty H, Das G and Ray B K 2011 *J. Orissa Math. Soc.* **30** (2) 13–34.
- [6] Osikiewicz J A 2000 *Analysis* **20** (1) 35-43.
- [7] Leindler L 2005 *Mathematical Analysis and Application* **302** (1) 129-136.
- [8] Mittal M L, Rhoades B E, Sonker S and Singh U 2011 *Appl.Math.Comp.* **217** (9) 4483-4489.
- [9] Nayak L, Das G and Ray B K 2014 *J. Math. Anal. Appl.* **420** (1) 563-575.
- [10] Nayak L, Das G and Ray B K 2014 *Int. J. Anal.* **2014** 171675.
- [11] Sonker S and Munjal A 2016 *Int. J. Math. Anal.* **10** (23) 1129-1136.
- [12] Sonker S and Munjal A 2016 *Nonlinear Studies* **23** (4) 533-542.
- [13] Sonker S and Munjal A 2017 *Nonlinear Studies* **24** (1) 113-125.
- [14] Sonker S and Munjal A 2017 *Proceedings ICAST-2017 Type A* **67**, 208-210.
- [15] Sonker S and Munjal A 2017 *Journal of Inequalities and Applications* **2017** (168) 1-7.
- [16] Sonker S and Munjal A 2017 *Int.J. Engg. Tech.* **9** (3S) 457-462.
- [17] Lal S and Nigam H K 2001 *Tamkang Journal of Mathematics* **32** 143–149.
- [18] Deger U, Dagadur I and Kucukaslan M 2012 *Proc. Jangjeon Math. Soc.* **15** (2) 203-213.
- [19] Degree of Approximation by a General C_λ - Summability Method