

# Validation of Analytical Damping Ratio by Fatigue Stress Limit

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**Abstract.** The optimisation process of a vibration energy harvester is usually restricted to experimental approaches due to the lack of an analytical equation to describe the damping of a system. This study derives an analytical equation, which describes the first mode damping ratio of a clamp-free cantilever beam under harmonic base excitation by combining the transverse equation of motion of the beam with the damping-stress equation. This equation, as opposed to other common damping determination methods, is independent of experimental inputs or finite element simulations and can be solved using a simple iterative convergence method. The derived equation was determined to be correct for cases when the maximum bending stress in the beam is below the fatigue limit stress of the beam. However, an increasing trend in the error between the experiment and the analytical results were observed at high stress levels. Hence, the fatigue limit stress was used as a parameter to define the validity of the analytical equation.

## 1. Introduction

The need for a sustainable energy source to power small electronics has caused an increase in research on energy harvesting. The idea of harnessing energy from ambient vibrations to power wireless sensors have been discussed in several literatures over the past decade [1–3]. The mechanism of a typical vibration energy harvester usually involves a cantilever beam with one end clamped to a vibrating structure; hence, subjecting the beam to base-excitation type motion. There are several approaches to convert the mechanical energy from the vibrating beam into electrical power, such as using piezoelectric transducers or through electromagnetic induction [4].

Optimisation of vibration energy harvesters have been the interest of most recent literatures [5–8]. The optimisation process generally involves design modifications in an attempt to produce more power or increase the frequency bandwidth of the device, while maintaining minimum size. Nevertheless, nearly all of the optimisation work involves experimental works or complex finite element algorithms



due to the lack of analytical equations to fully describe the vibration system, particularly the damping parameter.

Often, the equations developed for a vibration system requires the damping parameter as an input. Previous work demonstrated a mathematical model equation describing the motion of a clamp-free cantilever beam under harmonic base excitation [9]. While this equation was proven to match closely with the experimental results, the equation was not fully analytical in terms that the damping parameter is expected to be obtained either from experiment or literature. Since damping varies according to the material and the dimensions of a cantilever beam, experimental input is preferred for better accuracy. This tells us that the mathematical model itself is not able to predict how the cantilever beam would behave prior to the experiment if an accurate damping ratio of the cantilever beam is unknown. Hence, the development of an analytical equation to accurately describe damping is highly necessary as this would allow for better hands-off predictions on the behaviour of the vibrating object and open up new paths to optimisation [10]. In an earlier work, an attempt was made to predict the damping of a material based on the stress distribution function of a cantilever beam during vibration [11]. Based on the study, an ideal damping equation was proposed, which relates the maximum bending stress and fatigue limit stress of a structure to its loss factor. However, this equation assumes the maximum bending stress as an experimental input.

In this work, an analytical equation describing the first mode damping ratio of a cantilever beam was derived by combining the equation of motion for the transverse vibration of a cantilever beam with a refined damping-stress equation [9,12]. The analytical equation can be used to predict the damping ratio of any clamp-free cantilever beam under harmonic base-excitation, provided that the beam's material properties and dimensions are known. After comparing the analytical results with the experimental findings, the importance of the fatigue limit stress on the quality of the analytical results was discussed and the validity of the equation was summarised.

## 2. Derivation of analytical equation



**Figure 1.** Cantilever beam under harmonic base excitation.

Figure 1 illustrates the movement of a vibrating clamp-free cantilever beam under harmonic base excitation. In general, the transverse motion of the clamp-free beam in Figure 1 at position  $x$  and time  $t$  can be described by equation (1).

$$a(x, t) = z(x, t) + y(t) \quad (1)$$

where  $a(x, t)$  is the total amplitude of the vibrating beam,  $z(x, t)$  is the amplitude of the beam relative to its clamped base and  $y(t)$  is the harmonic base excitation amplitude. Based on the Euler-Bernoulli beam theory, the equation of motion for an undamped cantilever beam subjected to external force can be modelled using equation (2).

$$EI \frac{d^4 z(x, t)}{dx^4} + \rho A \frac{d^2 z(x, t)}{dt^2} = F(t) \quad (2)$$

where  $E$  is the Young's modulus of the cantilever beam,  $I$  is the second moment of area,  $\rho$  is the density,  $A$  is the beam's cross-sectional area and  $F(t)$  is the forcing function distributed over the length of the beam. For cases of base excitation problem, the forcing function  $F(t)$  signifies the driving force input of the base motion. Using the method of separation of variables, the term  $z(x, t)$  can be separated into its spatial and temporal components.

$$z(x, t) = \sum_{n=1}^{\infty} \varphi_n(x) \eta_n(t) \quad (3)$$

where  $\varphi_n(x)$  is the cantilever beam's modal-shape eigenfunction and  $\eta_n(t)$  is the regular-response function. The subscript  $n$  in equation 3 denotes the correspondence to the  $n^{th}$  mode of vibration. The cantilever beam's eigenfunction can be described as [9,13]

$$\varphi_n(x) = \left(\frac{1}{m}\right)^{1/2} \left[ \cosh \frac{\beta_n}{L} x - \cos \frac{\beta_n}{L} x - \frac{\sinh \beta_n - \sin \beta_n}{\cosh \beta_n + \cos \beta_n} \left( \sinh \frac{\beta_n}{L} x - \sin \frac{\beta_n}{L} x \right) \right] \quad (4)$$

where  $m$  is the mass of the unclamped part of the beam,  $L$  is the length of the beam and  $\beta_n$  is a dimensionless constant. The regular response function is the solution to the following second-order differential equation. It is easy to notice that equation (5) can be solved by referring to a single degree of freedom vibration problem.

$$\ddot{\eta}_n(t) + 2\zeta_n \omega_n \dot{\eta}_n(t) + \omega_n^2 \eta_n(t) = F(t) \quad (5)$$

where  $\dot{\eta}_n$  and  $\ddot{\eta}_n$  is the first and second derivative of the response function with respect to  $t$ ,  $\zeta_n$ , which is the cantilever beam modal damping ratio and  $\omega_n$  is the  $n^{th}$  mode natural frequency of the beam. The forcing function term can be expressed as [8].

$$F(t) = \frac{-m\omega^2 y e^{i\omega t}}{L} \int_0^L \varphi_n(x) dx \quad (6)$$

where  $\omega$  is the frequency of the harmonic driving force input. Solving equation (7) and substituting the solution into equation (6), the solution for equation (6) for cases of harmonic motion can be obtained.

$$\eta_n(t) = \frac{2\omega^2 m^{1/2} (\sinh \beta_n - \sin \beta_n)}{\beta_n (\omega_n^2 - \omega^2 + i2\zeta_n \omega_n \omega) (\cosh \beta_n + \cos \beta_n)} y e^{i\omega t} \quad (7)$$

Substituting Eqs. (7) and (4) into equation (3) and considering cases of first mode resonance where  $\omega = \omega_n$  and  $n = 1$ , the real part of equation (3) becomes

$$z(x, t) = y e^{i\omega_1 t} \left[ \cosh \frac{\beta_1}{L} x - \cos \frac{\beta_1}{L} x - \frac{\sinh \beta_1 - \sin \beta_1}{\cosh \beta_1 + \cos \beta_1} \left( \sinh \frac{\beta_1}{L} x - \sin \frac{\beta_1}{L} x \right) \right] \frac{\sinh \beta_1 - \sin \beta_1}{\beta_1 \zeta_1 (\cosh \beta_1 + \cos \beta_1)} \quad (8)$$

This accuracy of equation (8) has been validated by previous authors [9]. Applying the Euler-Bernoulli theory and Hooke's law, the stress acting on a cantilever beam can be described by the following equation

$$\sigma(x, t) = \frac{Eb}{2} \left( \frac{d^2 z}{dx^2} \right) \quad (9)$$

where  $\sigma(x, t)$  is the stress acting on the beam,  $b$  is the thickness of the beam and  $\frac{d^2 z}{dx^2}$  is the second derivative for the beam's vertical motion equation with respect to  $x$ . Substituting equation (8) into equation (9) and evaluating the beam at  $x = 0$  and  $ye^{i\omega_1 t} = y$ , the expression for the maximum bending stress of a vibrating beam can be obtained.

$$\sigma_m = Eby\beta_1 \frac{\sinh\beta_1 - \sin\beta_1}{L^2\zeta_1(\cosh\beta_1 + \cos\beta_1)} \quad (10)$$

where  $\sigma_m$  is the maximum bending stress experienced by the beam. Equation (10) relates the maximum bending stress on a beam to its damping ratio. Equation (11) describes another equation relating these two variables. The equation is described as refined damping-stress equation that relates the maximum stress and fatigue limit stress of a cantilever beam to its first mode loss factor [12].

$$\sigma_m = Eby\beta_1\gamma_1 = E \left( 2130.6 \frac{\sigma_m^{0.3}}{\sigma_f^{2.3}} + 8176.7 \frac{\sigma_m^6}{\sigma_f^8} \right) \quad (11)$$

where  $\gamma_1$  is the first mode loss factor and  $\sigma_f$  is the fatigue limit stress of the beam. In general, the loss factor of a cantilever beam is twice the value of its damping ratio. Taking this into account and substituting equation (10) into equation (11), the analytical equation for the first mode damping ratio can be derived in the form of

$$\zeta_1 = \frac{E}{2} \left[ 2130.6 \frac{\left( \frac{2Eby\beta_1 K_1}{L^2\zeta_1} \right)^{0.3}}{\sigma_f^{2.3}} + 8176.7 \frac{\left( \frac{2Eby\beta_1 K_1}{L^2\zeta_1} \right)^6}{\sigma_f^8} \right] \quad (12)$$

where

$$K_1 = \frac{\sinh\beta_1 - \sin\beta_1}{\cosh\beta_1 + \cos\beta_1} \quad (13)$$

The dimensions and material properties of a cantilever beam are usually known. Hence, the only unknown parameter in equation (12) is the damping ratio of the beam. Since this unknown parameter appears on both sides of the equation, equation (12) can be solved using an iterative converging algorithm, such as the secant method or the Newton-Raphson method.

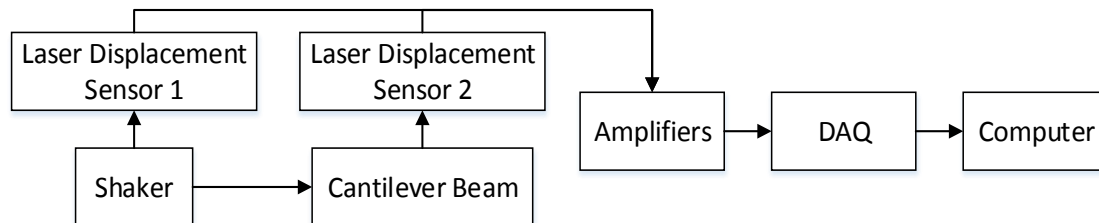
### 3. Results and Discussion

In this section, the analytical equation from equation (12) was compared to the experimental results. An aluminium and a steel cantilever beam were used in the experiments; their properties are tabulated in Table 1 below.

**Table 1.** Specifications of the specimen cantilever beams.

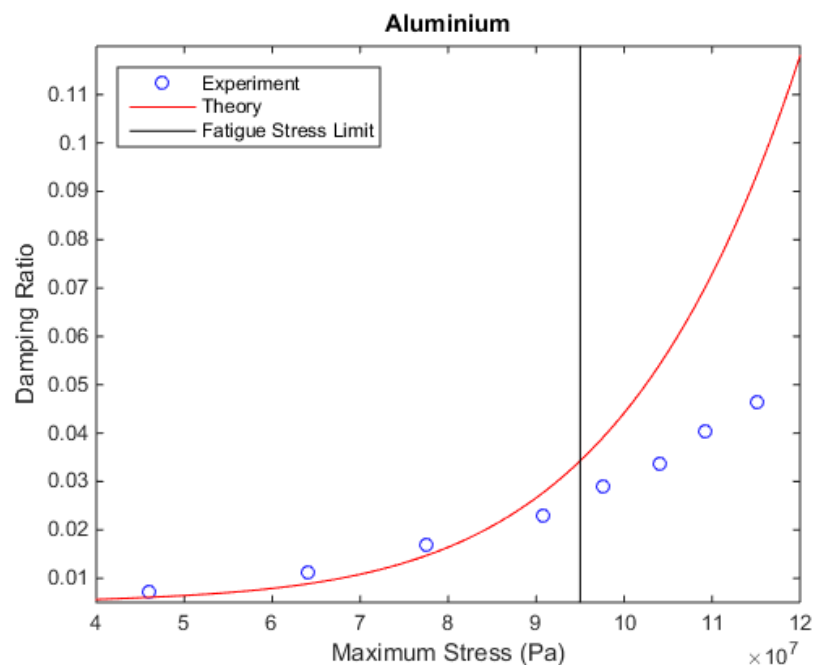
Material	$E$ (GPa)	$t$ (mm)	$w$ (mm)	$\sigma_f$ (Mpa)
Aluminium	60	1.20	18.14	95
Steel	190	0.50	12.75	310

The experiment involved subjecting a cantilever beam to a harmonic base-excitation vibration by clamping one end of the beam onto an electromagnetic shaker. The output response of the beam and the shaker was recorded using laser displacement sensors and the damping ratio was determined based on these responses. Figure 2 shows the schematic of the experimental setup.

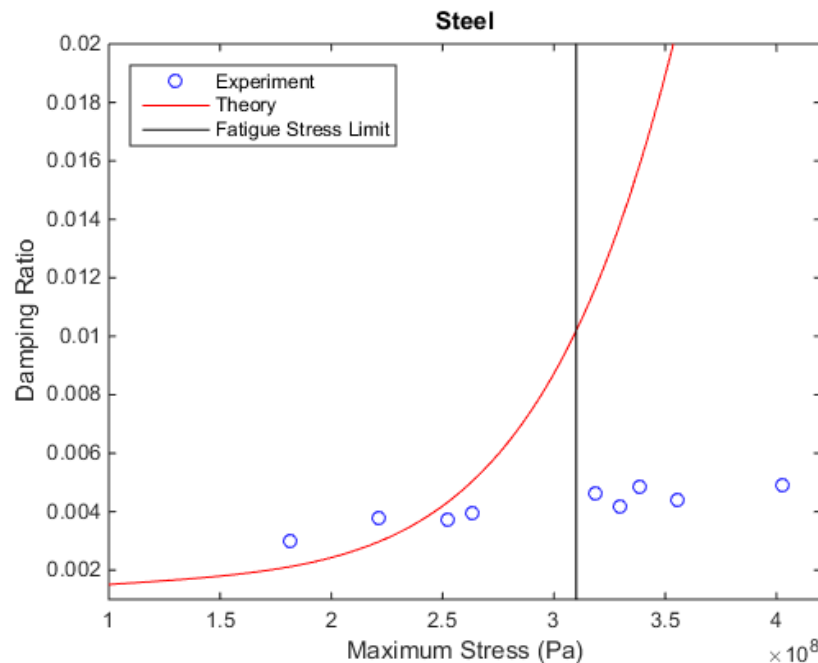


**Figure 2.** Schematic of the experimental setup used to determine the damping ratio of cantilever beams.

For each beam, the experiment was repeated several times to induce different damping ratios by changing the length of the beam or the base excitation amplitude. The thickness and the width of the beam remained fixed. Usually, the preferred method to determine the damping ratio of a beam specimen under base excitation cases is to use the half-power bandwidth method. However, this method can lead to inconsistent results as the damping ratio obtained is highly dependent on the quality of the frequency response curve. Hence, for a more substantial result, the experimental damping ratio was calculated by using equation (8), where the maximum relative amplitude at  $x = L$  and the corresponding base excitation amplitude was obtained from the experiment. The maximum bending stress at the clamped end of the beam was calculated using equation (10) based on the determined damping ratio. Figure 3 and Figure 4 illustrate the comparison between the analytical and the experimental results.



**Figure 3.** Comparison between experimental and analytical results for aluminium cantilever beam.



**Figure 4.** Comparison between experimental and analytical results for steel cantilever beam.

The results demonstrated that the experimental findings closely correlated to the analytical equation at low stress values, specifically below the fatigue limit stress of the beam. The maximum error recorded between the analytical and the experimental damping results at stresses below the fatigue limit stress for aluminium is approximately 21%, whereas for steel it is around 29%. The errors in both beams are seen to increase as the stresses exceed the fatigue limit stress, although steel recorded a more significant error with a maximum reaching 735%. It is easily noticed that although a large error was recorded for both beams at stresses above the fatigue stress limit, the damping ratio recorded at these stresses still displayed an increasing trend. This is in agreement to what was discussed in earlier studies in where the damping of a structure is highly dependent on the stress distribution of the structure [11]. The errors recorded in this work may be due to the fact that the damping-stress equation in equation (11) was a refinement of another damping-stress equation, which was described to be an idealised relation generalising high non-linear stress behaviour [11,12]. In reality, several other different types of damping-stress curve representations were also recorded in his work. Nevertheless, if the stress levels were constrained to values below the fatigue stress limit, the analytical equation can be concluded to provide a good estimation on the damping ratio of any cantilever beam. This, in turn, defines the validity of the analytical equation, where it is only valid within the mentioned constraints.

In terms of vibration energy harvesting application, current research focuses on design miniaturization as these devices are generally targeted to be installed in small electronics. Hence, the vibrating beam is not expected to vibrate at high stresses due to the volume constraint and would almost definitely not reach the fatigue stress limit. Therefore, it is possible to optimise these small devices and expect a fairly good prediction using the analytical equation. In general, the analytical equation can provide a good insight to designers or material scientists on the damping capacity of different materials, where instead of spending a large sum of money on material testing, one can apply the equation to have a broad idea on the average damping capacity of a material.

#### 4. Conclusion

An analytical equation describing the first mode damping ratio of a clamp-free cantilever beam under base-excitation motion was proposed and the importance of the fatigue limit stress on the proposed

equation was discussed. The analytical equation displayed a good agreement with experimental results when the cantilever beam experienced low stress levels. However, large errors were recorded when the maximum stress exceeded the fatigue limit stress of the beam, with the errors increasing further as the maximum stress in the beam increased. This limitation describes the validity of the analytical equation, where it was concluded that the equation would only be valid for cases where the maximum bending stress is below the fatigue limit stress. The work presented here does not account for cases where a lumped mass is placed at the free end tip of the cantilever beam. Hence, future plans include deriving the damping ratio equation for cases of cantilever beams with lumped mass and investigating the accuracy of this equation.

### Acknowledgment

The authors would like to thank to Fundamental Research Grant Scheme (FRGS) from Ministry of Higher Education (MOHE) Malaysia, Grant No: FRGS/1/2016/TK03/HWUM/03/1, for funding this research.

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