

A study on a Ivlev-type Chemostat model with one competitor producing a toxin

Mingjuan Sun*

School of Mathematics and Computer Science, Yanan University, Yanan, China

*Corresponding author e-mail: sunmingjuan0535@163.com

Abstract. In order to illustrate the phenomena microorganisms produce toxins to compete with its competitors, a Chemostat model of two competitors for single nutrient with Ivlev-type functional response is investigated. The existence of its solutions is analysed, and the local asymptotic stability of the model's equilibria is obtained. By use of the theory of limit systems and Lyapunov-LaSalle invariant principle, it is shown that its equilibria are globally asymptotically stable under some conditions. Our results demonstrate that the principle of competition exclusion is established, and under different conditions, different microorganism can beat the competition.

1. Introduction

A Chemostat is a common laboratory apparatus which can be used to culture micro-organisms^[1,2], for example, it can be used to model the simple lake in ecology, and it can also be used to model the bioreactor which is the apparatus of manufacturing products in chemical engineering. The basic exploitative competition Chemostat model consists of equations of two microorganisms and single nutrient. In nature, however, in order to gain survival advantage, micro-organisms can produce inhibitors (toxins) to compete with their rivals^[3-6]. Therefore, we also need an equation to describe the toxins.

It is known that functional response is an important factor in the modelling of Chemostat, and there are many different types of functions which have been used in the literature, for example, Holling-type, Tissiet-type, Ratio-dependent-type, Crowley–Martin-type, Ivlev-type functional response, and so on^[1]. And it has been proved that they are important and suitable to describe different phenomena in practice. In this article, the Ivlev-type function will be used to account for the functional response, and it is an efficient tool in the modelling of Chemostat and predator-prey systems. Wang et al.^[7] constructed a reaction–diffusion predator–prey model and gave detailed analysis of Hopf bifurcation, where the Ivlev-type functional response is used. Using the Ivlev functional response, Gong et al.^[8] studied a stochastic single-species Chemostat system and obtained the threshold conditions of the persistence in mean and the extinction of microorganisms. Considering the disease in prey, Shao et al.^[9] proposed a predator-prey model with Ivlev-type functional response and obtained the permanence of the system. Liu et al.^[10] studied the dynamics properties of a predator–prey model with Ivlev functional response and time delay, they showed that there is a Hopf bifurcation. Base on the above discussion, the Ivlev functional response theory may work in the Chemostat model to describe the phenomena that one of the competitors may produces toxins to compete with its rivals. To author's best knowledge, considering that one competitor may produces a toxin, the Chemostat model with Ivlev-type functional response has not been found. In the paper, the following system is discussed.



$$\begin{cases} S' = 1 - S - m_1 x (1 - e^{-c_1 S}) e^{-\mu P} - m_2 y (1 - e^{-c_2 S}) := F_1(S, x, y, P), \\ x' = x \left[m_1 (1 - e^{-c_1 S}) e^{-\mu P} - 1 \right] := F_2(S, x, y, P), \\ y' = y \left[(1 - k) m_2 (1 - e^{-c_2 S}) - 1 \right] := F_3(S, x, y, P), \\ P' = k m_2 y (1 - e^{-c_2 S}) - P := F_4(S, x, y, P). \end{cases} \quad (1)$$

where $x(t)$, $y(t)$ and $S(t)$ denote concentrations of the microorganisms and the nutrient at time t , respectively, $P(t)$ denotes the concentration of toxin at present, $\mu_1(S) = m_1(1 - e^{-c_1 S})$ and $\mu_2(S) = m_2(1 - e^{-c_2 S})$ are the Ivlev-type functional response functions, and m_1 , m_2 , c_1 and c_2 are all positive constants. With the same discussion as [3], in this paper, we assume $0 < k < 1$. Moreover, $e^{-\mu P}$ denotes the effect of toxin on the growth of microorganisms, which is different from that was discussed in [3].

From the biological point of view, the initial conditions of (1) are

$$S(0) = S_0, \quad x(0) = x_0 \geq 0, \quad y(0) = y_0 \geq 0, \quad P(0) = P_0 \geq 0. \quad (2)$$

The remainder of the article is organized as follows. Some preliminary results including existence of the equilibria of (1) and ultimate boundedness of the solutions of (1) are given in section 2. In section 3, it is shown that its equilibria are globally asymptotically stable under some conditions. Section 4 concludes the paper.

2. Preliminary analysis

In this section, we obtain some basic results about the positively invariant set and the equilibria are presented.

Theorem 2.1 The set $\Omega = \{(x, y) \mid 0 \leq S \leq 1, x \geq 0, y \geq 0, P \geq 0\}$ is positively invariant with respect to (1) and attracts any solution of (1), moreover

$$\lim_{t \rightarrow \infty} (S(t) + x(t) + y(t) + P(t)) = 1. \quad (3)$$

Proof. Using the same method as [6], the result is obtained easily. ■

Theorem 2.2 For the existence of the equilibria, we have

(i) There is always a washout equilibrium $E_1(1, 0, 0, 0)$;

(ii) When $m_2 > e^{c_2} / ((e^{c_2} - 1)(1 - k))$, there is a boundary equilibrium

$$E_2 \left(\frac{1}{c_2} \ln \left(\frac{(1 - k)m_2}{(1 - k)m_2 - 1} \right), 0, (1 - k) \left(1 - \frac{1}{c_2} \ln \left(\frac{(1 - k)m_2}{(1 - k)m_2 - 1} \right) \right), k \left(1 - \frac{1}{c_2} \ln \left(\frac{(1 - k)m_2}{(1 - k)m_2 - 1} \right) \right) \right);$$

(iii) When $m_1 > e^{c_1} / (e^{c_1} - 1)$, there is a boundary equilibrium

$$E_3 \left(\frac{1}{c_1} \ln \left(\frac{m_1}{m_1 - 1} \right), 1 - \frac{1}{c_1} \ln \left(\frac{m_1}{m_1 - 1} \right), 0, 0 \right);$$

(iv) When $m_1 \left(1 - \left(\frac{(1 - k)m_2 - 1}{(1 - k)m_2} \right)^{c_1/c_2} \right) > 1$, there is a boundary equilibrium

$$E_4 \left(S^*, 1 - S^* - \frac{1 - k}{\mu k} \ln \left(m_1 (1 - e^{-c_1 S^*}) \right) - \frac{1}{\mu} \ln \left(m_1 (1 - e^{-c_1 S^*}) \right), \frac{1 - k}{\mu k} \ln \left(m_1 (1 - e^{-c_1 S^*}) \right), \frac{1}{\mu} \ln \left(m_1 (1 - e^{-c_1 S^*}) \right) \right),$$

$$\text{where } S^* = \frac{1}{c_2} \ln \left(\frac{(1 - k)m_2}{(1 - k)m_2 - 1} \right).$$

Proof. We can complete the proof of the existence by solving equations $F_i(S_i^*, x_i^*, y_i^*, P_i^*) = 0$, $i = 1, 2, 3, 4$. Firstly, the washout solution $E_1(1, 0, 0, 0)$ can be obtained easily. Secondly, for the boundary equilibrium E_2 , from $(1-k)m_2(1-e^{-c_2 S_2^*}) = 1$, the expression of S_2^* is obtained, if $m_2 > e^{c_2} / ((e^{c_2} - 1)(1-k))$. It follows from $F_1(S_2^*, x_2^*, y_2^*, P_2^*) = 0$ and $F_4(S_2^*, x_2^*, y_2^*, P_2^*) = 0$, $y_2^* = (1-k)(1-S_2^*)$ and $P_2^* = k(1-S_2^*)$ are obtained. Similarly, we can get E_3 and E_4 . ■

3. Global asymptotic stability analysis

In this section, let us consider global asymptotic stability of the equilibria of (1). From Theorem 2.1, it is enough to consider (1) on Ω . For the asymptotical stability of the equilibria, we have the following theorems.

Theorem 3.1 (i) When $m_1(1-e^{-c_1}) < 1$ and $(1-k)m_2(1-e^{-c_2}) < 1$, E_1 is locally asymptotically stable; (ii) When $m_2 > e^{c_2} / ((e^{c_2} - 1)(1-k))$ and $m_1(1-e^{-c_1 S_2^*})e^{-\mu P_2^*} < 1$, E_2 is locally asymptotically stable; (iii) When $m_1 > e^{c_1} / (e^{c_1} - 1)$ and $(1-k)m_2 \left(1 - \left(\frac{m_1-1}{m_1} \right)^{c_2/c_1} \right) < 1$, E_3 is locally asymptotically stable; (iv) When E_4 exists, it is unstable.

Proof. Let $E_i = (S_i^*, x_i^*, y_i^*, P_i^*)$, $i = 1, 2, 3, 4$ be the equilibrium of (1). By simple computations, it easily has that Jacobian matrix $J(\bar{E})$ of (1) at $\bar{E}$ as follows

$$J(\bar{E}) = \begin{pmatrix} -1 - m_1 c_1 \bar{x} e^{-\mu \bar{P}} e^{-c_1 \bar{S}} - m_2 c_2 y e^{-c_1 \bar{S}} & -m_1(1-e^{-c_1 \bar{S}})e^{-\mu \bar{P}} & -m_2(1-e^{-c_1 \bar{S}}) & \mu m_1 \bar{x}(1-e^{-c_1 \bar{S}})e^{-\mu \bar{P}} \\ m_1 c_1 \bar{x} e^{-\mu \bar{P}} e^{-c_1 \bar{S}} & m_1(1-e^{-c_1 \bar{S}})e^{-\mu \bar{P}} - 1 & 0 & -\mu m_1 \bar{x}(1-e^{-c_1 \bar{S}})e^{-\mu \bar{P}} \\ (1-k)m_2 c_2 \bar{y} e^{-c_1 \bar{S}} & 0 & (1-k)m_2(1-e^{-c_1 \bar{S}}) - 1 & 0 \\ km_2 c_2 \bar{y} e^{-c_1 \bar{S}} & 0 & km_2(1-e^{-c_1 \bar{S}}) & -1 \end{pmatrix}$$

For E_1 , we can get the characteristic equation

$$(\lambda + 1)^2 (\lambda - m_1(1-e^{-c_1}) + 1)(\lambda - (1-k)m_2(1-e^{-c_2}) + 1) = 0,$$

thus, the characteristic roots are $\lambda_{1,2} = -1$, $\lambda_3 = m_1(1-e^{-c_1}) - 1$ and $\lambda_4 = (1-k)m_2(1-e^{-c_2}) - 1$.

Hence, when $m_1(1-e^{-c_1}) < 1$ and $(1-k)m_2(1-e^{-c_2}) < 1$, E_1 is locally asymptotically stable.

Similarly, for E_2 , we can get the characteristic roots $\lambda_{1,2} = -1$, $\lambda_3 = m_1(1-e^{-c_1 S_2^*})e^{-\mu P_2^*} - 1$ and $\lambda_4 = -m_2 y_2^* e^{-c_2 S_2^*}$. Hence, when $m_2 > e^{c_2} / ((e^{c_2} - 1)(1-k))$ and $m_1(1-e^{-c_1 S_2^*})e^{-\mu P_2^*} < 1$, E_2 is locally asymptotically stable.

For E_3 , we can get the characteristic roots $\lambda_{1,2} = -1$, $\lambda_3 = (1-k)m_2(1-e^{-c_2 S_3^*}) - 1$ and

$\lambda_4 = -m_1 c_1 x_3^* e^{-c_1 S_3^*}$. Hence, when $m_1 > e^{c_1} / (e^{c_1} - 1)$ and $(1-k)m_2 \left(1 - \left(\frac{m_1-1}{m_1} \right)^{c_2/c_1} \right) < 1$, E_3 is

locally asymptotically stable.

For E_4 , we can get the characteristic roots $\lambda_{1,2} = -1$, λ_3 and λ_4 satisfy the following equation

$$\lambda^2 + (m_1 c_1 x_4^* e^{-\mu P_4^*} e^{-c_1 S_4^*} + m_2 c_2 y_4^* e^{-c_2 S_4^*}) \lambda - \mu k m_2 c_2 x_4^* y_4^* e^{-c_2 S_4^*} = 0, \quad (4)$$

and $\lambda_3 \lambda_4 = -\mu k m_2 c_2 x_4^* y_4^* e^{-c_2 S_4^*} < 0$, therefore, E_4 is unstable, when it exists. ■

- Theorem 3.2** (i) E_1 is globally asymptotically stable for $m_1(1 - e^{-c_1}) < 1$ and $(1 - k)m_2(1 - e^{-c_2}) < 1$;
 (ii) E_2 is globally asymptotically stable for $m_2 > e^{c_2} / ((e^{c_2} - 1)(1 - k))$ and $m_1(1 - e^{-c_1 S_2^*})e^{-\mu P_2^*} < 1$;
 (iii) E_3 is globally asymptotically stable for $m_1 > e^{c_1} / (e^{c_1} - 1)$ and $(1 - k)m_2 \left(1 - \left(\frac{m_1 - 1}{m_1} \right)^{c_2/c_1} \right) < 1$.

Proof. By use of Lyapunov-LaSalle invariance principle^[6], it is first shown that E_1 is globally attractive. Consider $V = x + y$ on the compact set Ω . Clearly, $V = x + y$ is continuous on $\Omega = \bar{\Omega}$ and it follows that

$$\begin{aligned} \dot{V}|_{(1)} = \dot{x}(t) + \dot{y}(t) &= x \left[m_1 (1 - e^{-c_1 S}) e^{-\mu P} - 1 \right] + y \left[(1 - k)m_2 (1 - e^{-c_2 S}) - 1 \right] \\ &\leq x \left[m_1 (1 - e^{-c_1}) - 1 \right] + y \left[(1 - k)m_2 (1 - e^{-c_2}) - 1 \right] \end{aligned} \quad (5)$$

which implies that we obtain a Lyapunov function of (1) on Ω , i.e., $V = x + y$. Next, the subset G of Ω is defined as follows

$$G = \{(x, y) \mid (x, y) \in \Omega = \bar{\Omega}, \dot{V} = 0\} , \quad (6)$$

and let M be the largest invariant set of (1) in Ω . Since $m_1(1 - e^{-c_1}) < 1$ and $(1 - k)m_2(1 - e^{-c_2}) < 1$, it has that

$$G = \{(x, y) \mid (x, y) \in \Omega, \dot{V} = 0\} = \{(x, y) \mid (x, y) \in \Omega, x = 0, y = 0\} .$$

By the invariance of M and (1), it easily has that $M = \{E_1\}$. Therefore, it follows from the well-known Lyapunov-LaSalle invariance principle that E_1 is globally attractive.

Now the globally asymptotical stability of E_2 is proved with the theory of limit systems. Since $S \leq 1$ and $P \geq 0$, we have $\lim_{t \rightarrow \infty} x(t) = 0$ for $m_1(1 - e^{-c_1}) < 1$. Thus, it is enough to consider system (1)'s limit system

$$\begin{cases} S' = 1 - S - m_2 y (1 - e^{-c_2 S}) := M(S, y), \\ y' = y \left[(1 - k)m_2 (1 - e^{-c_2 S}) - 1 \right] := N(S, y), \end{cases} \quad (7)$$

and

$$P' = km_2 y (1 - e^{-c_2 S}) - P .$$

Since there does not exist the variable $P(t)$ in the equations (7), it is sufficient to consider the behaviour of (7). Let us choose $B(x, y) = y^{-1}(1 - S)^{-1}$ as a Dulac function for (7). Hence, it has that

$$\frac{\partial(BM)}{\partial S} + \frac{\partial(BN)}{\partial y} = -\frac{m_2 c_2 e^{-c_2 S}}{1 - S} - \frac{m_2}{(1 - S)^2} < 0 \quad (8)$$

for $(S, y) \in \Omega$. It follows from Dulac criterion that there cannot exist any closed trajectories in Ω . Furthermore, from Theorem 3.1, E_2 is locally asymptotically stable for $m_2 > e^{c_2} / ((e^{c_2} - 1)(1 - k))$ and $m_1(1 - e^{-c_1 S_2^*})e^{-\mu P_2^*} < 1$, thus, E_2 is globally asymptotically stable. Similarly, we can get the conclusion that E_3 is globally asymptotically stable. ■

4. Conclusions

In this article, based on some ideas in [3-6] etc, considering the effect of the toxins on the growth of microorganisms, a Chemostat model with Ivlev-type functional response is constructed. The global asymptotical stability of its equilibria is studied. From Theorem 3.2, under different conditions, the

global asymptotical stability of E_2 and E_3 are proved, respectively, that is to say, the principle of competitive exclusion is established, and under different conditions, different microorganism can beat the competition.

Acknowledgments

This work was supported by the Funds of YAU (YD2015-10, D2017026).

References

- [1] L. Chen and J. Chen. Nonlinear Biology Dynamics. Beijing: Science Press, 1993.
- [2] H. L. Smith and P. Waltman. The Theory of the Chemostat: Dynamics of Microbial Competition. Cambridge: Cambridge University Press, 1995.
- [3] Hsu S B, Waltman P. Competition in the chemostat when one competitor produces a toxin. Japan Journal of Industrial and Applied Mathematics 15 (1998) 471-490.
- [4] Huang X, Wang Y, Zhu L. Competition in the bioreactor with general quadratic yields when one competitor produces a toxin. Journal of Mathematical Chemistry 39 (2006) 281-294.
- [5] Li J, Yang Y, Yang G. Qualitative analysis of a chemostat model with production of toxin. Journal of engineering mathematics 24 (2007) 365-368.
- [6] Sun M, Dong Q, Dai L. Qualitative analysis of a ratio-dependent Chemostat model with production of toxin. Journal of System Science and Mathematical Science 37 (2107) 2018-2027.
- [7] Wang X, Wei J. Dynamics in a diffusive predator-prey system with strong Allee effect and Ivlev-type functional response. Journal of Mathematical Analysis & Applications 422 (2015) 1447-1462.
- [8] Gong C, Xu C, Yuan S. Threshold of a stochastic Chemostat model with the Ivlev functional response function. Journal of University of Shanghai for Science and Technology 39 (2017) 1-6.
- [9] Shao Y, Li P, Tang G. Dynamic analysis of an impulsive predator-prey model with disease in prey and Ivlev-type functional response. Abstract & Applied Analysis 2012 (2015) 1-16.
- [10] Liu W, Jiang Y, Chen Y. Dynamic properties of a delayed predator-prey system with Ivlev-type functional response. Nonlinear Dynamics 84 (2016) 743-754.