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A New Theory of Non-Linear Thermo-Elastic Constitutive Equation of Isotropic Hyperelastic Materials

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Abstract. Considering the influence of temperature and strain variables on materials. According to the relationship of conjugate stress-strain, a complete and irreducible non-linear constitutive equation of isotropic hyperelastic materials is derived and the constitutive equations of 16 types of isotropic hyperelastic materials are given we study the transformation methods and routes of 16 kinds of constitutive equations and the study proves that transformation of two forms of constitutive equation. As an example of application, the non-linear thermo-elastic constitutive equation of isotropic hyperelastic materials is combined with the natural vulcanized rubber experimental data in the existing literature base on MATLAB, The results show that the fitting accuracy is satisfactory.

1. Introduction

Hyperelastic materials refer to rubber materials, polymers, synthetic elastomers and other biomaterials such as blood vessels, skin and muscle.Because of its unique mechanical properties, hyperelastic materials are widely used in aerospace, mechanical manufacturing, biological medicine and other fields.Some studies indicate that the mechanical properties of hyperelastic materials are sensitive to temperature.The thermal constitutive equation of hyperelastic materials considering temperature influence has become a hot topic in many scholars[1-7].The Thermo-constitutive Equation is not only a precondition for the numerical simulation of complex thermal structure, but also an important theoretical basis for further study of the relationship between the rheological stress and temperature and strain rate of the materials.Therefore, it is of great significance to establish accurate thermal constitutive model for the study of hyperelastic materials.At present, the constitutive equation which consider the temperature influence always used the phenomenological analysis method[8-12].This paper adopts tensor method, consider the influence of temperature and strain variables on the constitutive of materials, a on-linear constitutive equation of is otropic hyperelastic materials is derived. A new theory of non-linear thermo-elastic constitutive equation of isotropic hyperelastic Materials is established. The constitutive equations of 16 different forms are derived, the non-linear thermo-elastic constitutive equation of isotropic hyperelastic materials is combined with the natural vulcanized rubber experimental data in the existing literature base on MATLAB, The results show that the fitting accuracy is satisfactory.

2. Non-linear Constitutive Equation of Isotropic Elastomer

We corresponded the stress strain relationship one by one, The pairs of variables are called conjugate stress strain variables. In continuum mechanics, we can define the following conjugate stress strain variables.

(1) $\mathbf{K}$ and $\mathbf{E}$ are conjugate stress strain variables.

(2) $\mathbf{K}$ and $\mathbf{c}$ are conjugate stress strain variables.
$\mathbf{c}$ is right Cauchy-Green strain tensor.

(3) $\mathbf{K}$ and $\mathbf{c}^{-1}$ are conjugate stress strain variables.

(4) $\mathbf{T}$ and $\mathbf{B}$, $\mathbf{B}^{-1}$, $\mathbf{e}$ are conjugate stress strain variables.

(5) $\mathbf{T}$ is Cauchy stress tensor. $\mathbf{B}$ is left Cauchy-Green strain tensor. $\mathbf{e}$ is Euler strain tensor.

(6) $\mathbf{S}$ and $\overline{\mathbf{U}}$, $\mathbf{U}$, $\mathbf{U}^{-1}$ are conjugate stress strain variables.
$\mathbf{S}$ is Jaumann stress tensor. $\mathbf{U}$ is right Cauchy—Green stretch tensor. $\overline{\mathbf{U}}$ is Elongation ratio tensor.

(7) $\mathbf{M}$ and $\mathbf{V}$, $\mathbf{V}^{-1}$, $\overline{\mathbf{V}}$ are conjugate stress strain variables.
$\mathbf{M}$ is Biot The symmetrical part of the stress. $\mathbf{V}$ is left Cauchy—Green stretch tensor. $\overline{\mathbf{V}}$ is left Elongation ratio tensor.

(8) Relationships between various strains

$$\begin{cases} \mathbf{c} = \mathbf{U}^2 \\ \overline{\mathbf{U}} = \mathbf{U} - \mathbf{1} \\ \mathbf{B} = \mathbf{V}^2 \\ \overline{\mathbf{V}} = \mathbf{1} - \mathbf{V} \\ \mathbf{c} - \mathbf{1} = 2\mathbf{E} \\ \mathbf{1} - \mathbf{B}^{-1} = 2\mathbf{e} \\ \mathbf{c} = \mathbf{R}^T \mathbf{B} \mathbf{R} \end{cases}$$

(9) Relationships between various stresses:

1. $J\mathbf{T} = \mathbf{FKF}^T$, J is Jacobian determinant. 3. $\mathbf{S} = \frac{1}{2}(\mathbf{K} g \mathbf{U} + \mathbf{U} g \mathbf{K})$.

3. $\mathbf{R} g \mathbf{U} g \mathbf{K} g \mathbf{R}^T = J\mathbf{V}^{-1}\mathbf{T}$. 4. $\mathbf{M} = \frac{1}{2} J \left(\mathbf{T} g \mathbf{V}^{-1} + \mathbf{V}^{-1} g \mathbf{T}\right)$.

According to the conjugate stress strain correspondence, we can be sure the constitutive equations of 16 types of isotropic elastic materials:

$$\mathbf{K}(\mathbf{E})\ \mathbf{K}(\mathbf{c})\ \mathbf{K}\left(\mathbf{c}^{-1}\right)\ \mathbf{K}\left(\mathbf{c}^{-1},\mathbf{c}\right);\ J\mathbf{T}(\mathbf{B})\ J\mathbf{T}\left(\mathbf{B}^{-1}\right)\ J\mathbf{T}\left(\mathbf{B}^{-1},\mathbf{B}\right)\ J\mathbf{T}(\mathbf{e})$$

$$\mathbf{S}\left(\overline{\mathbf{U}}\right)\ \mathbf{S}(\mathbf{U})\ \mathbf{S}\left(\mathbf{U}^{-1}\right)\ \mathbf{S}\left(\mathbf{U}^{-1},\mathbf{U}\right);\mathbf{M}\left(\overline{\mathbf{V}}\right)\ \mathbf{M}(\mathbf{V})\ \mathbf{M}\left(\mathbf{V}^{-1}\right)\ \mathbf{M}\left(\mathbf{V}^{-1},\mathbf{V}\right)$$

According theorem of representation,the constitutive equations of 16 types of isotropic elastic materials is derived. can be expressed as:

$$\begin{cases} \mathbf{K} = \phi_0 \mathbf{1} + \phi_1 \mathbf{c} + \phi_2 \mathbf{c}^2 \\ \mathbf{K} = \phi_0^{'} \mathbf{1} + \phi_1^{'} \mathbf{c}^{-1} + \phi_2^{'} \mathbf{c}^{-2} \\ \mathbf{K} = \phi_0^{''} \mathbf{1} + \phi_1^{''} \mathbf{c} + \phi_2^{''} \mathbf{c}^{-1} \end{cases} \qquad \begin{cases} J\mathbf{T} = \phi_0^{''} \mathbf{1} + \phi_1^{''} \mathbf{B} + \phi_2^{''} \mathbf{B}^2 \\ J\mathbf{T} = \beta_0^{'} \mathbf{1} + \beta_1^{'} \mathbf{B}^{-1} + \beta_2^{'} \mathbf{B}^{-2} \\ J\mathbf{T} = \beta_0^{''} \mathbf{1} + \beta_1^{''} \mathbf{B} + \beta_2^{''} \mathbf{B}^{-1} \\ J\mathbf{T} = \rho_0 \mathbf{1} + \rho_1 \mathbf{e} + \rho_2 \mathbf{e}^2 \end{cases}$$

$$\begin{cases} \mathbf{S} = \varsigma_0 \mathbf{1} + \varsigma_1 \overline{\mathbf{U}} + \varsigma_2 \overline{\mathbf{U}}^2 \\ \mathbf{S} = \psi_0 \mathbf{1} + \psi_1 \mathbf{U} + \psi_2 \mathbf{U}^2 \\ \mathbf{S} = \psi_0^{'} \mathbf{1} + \psi_1^{'} \mathbf{U}^{-1} + \psi_2^{'} \mathbf{U}^{-2} \\ \mathbf{S} = \psi_0^{''} \mathbf{1} + \psi_1^{''} \mathbf{U} + \psi_2^{''} \mathbf{U}^{-1} \end{cases} \qquad \begin{cases} \mathbf{M} = \zeta_0 \mathbf{1} + \zeta_1 \overline{\mathbf{V}} + \zeta_2 \overline{\mathbf{V}}^2 \\ \mathbf{M} = h_0 \mathbf{1} + h_1 \mathbf{V} + h_2 \mathbf{V}^2 \\ \mathbf{M} = h_0^{'} \mathbf{1} + h_1^{'} \mathbf{V}^{-1} + h_2^{'} \mathbf{V}^{-2} \\ \mathbf{M} = h_0^{''} \mathbf{1} + h_1^{''} \mathbf{V} + h_2^{''} \mathbf{V}^{-1} \end{cases}$$

The above 16 forms of constitutive equations can be transformed,there are three different forms of transformation be proved.

according to Theorem of representation,the constitutive equation $\mathbf{K} = \mathbf{K}(\mathbf{E})$ can be expressed as:

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$$\mathbf{K} = \varphi_0 \mathbf{1} + \varphi_1 \mathbf{E} + \varphi_2 \mathbf{E}^2 \tag{1}$$

$$\because \mathbf{E} = \frac{1}{2}(\mathbf{c} - \mathbf{1}) \qquad \therefore \mathbf{E}^2 = \frac{1}{4}(\mathbf{c}^2 - 2\mathbf{c} + \mathbf{1}) \tag{2}$$

Take Eq. (2) into Eq. (1), we can get:

$$\begin{aligned} \mathbf{K} &= \varphi_0 \mathbf{1} + 2\varphi_1[\frac{1}{2}(\mathbf{c} - \mathbf{1})] + 3\varphi_2[\frac{1}{4}(\mathbf{c}^2 - 2\mathbf{c} + \mathbf{1})] \\ &= (\varphi_0 - \frac{1}{2}\varphi_1 + \frac{1}{4}\varphi_2)\mathbf{1} + \frac{1}{2}(\varphi_1 - \varphi_2)\mathbf{c} + \frac{1}{4}\varphi_2 \mathbf{c}^2 \\ &= \phi_0 \mathbf{1} + \phi_1 \mathbf{c} + \phi_2 \mathbf{c}^2 \end{aligned} \tag{3}$$

Let $\mathbf{c}^{-1}$ multiply Cayley-Hamilton tensor Equation($\mathbf{c}^3 = I_1 \mathbf{c}^2 - I_2 \mathbf{c} + I_3 \mathbf{1}$)

we can get: $\mathbf{c}^2 = I_1 \mathbf{c} - I_2 \mathbf{1} + I_3 \mathbf{c}^{-1}$

Take $\mathbf{c}^2$ into Eq. (3) we can get:

$$\mathbf{K} = \phi_0' \mathbf{1} + \phi_1' \mathbf{c}^{-1} + \phi_2' \mathbf{c}^{-2} \tag{4}$$

Let $\mathbf{c}^{-1}$ multiply Cayley-Hamilton tensor Equation($\mathbf{c}^3 = I_1 \mathbf{c}^2 - I_2 \mathbf{c} + I_3 \mathbf{1}$)

we can get: $\mathbf{c} = I_1 \mathbf{1} - I_2 \mathbf{c}^{-1} + I_3 \mathbf{c}^{-2}$

Take $\mathbf{c}$ into Eq. (4) we can get:

$$\mathbf{K} = \phi_0'' \mathbf{1} + \phi_1'' \mathbf{c} + \phi_2'' \mathbf{c}^{-1} \tag{5}$$

The other equation of form can be proved with this method,the route of prove as this picture:

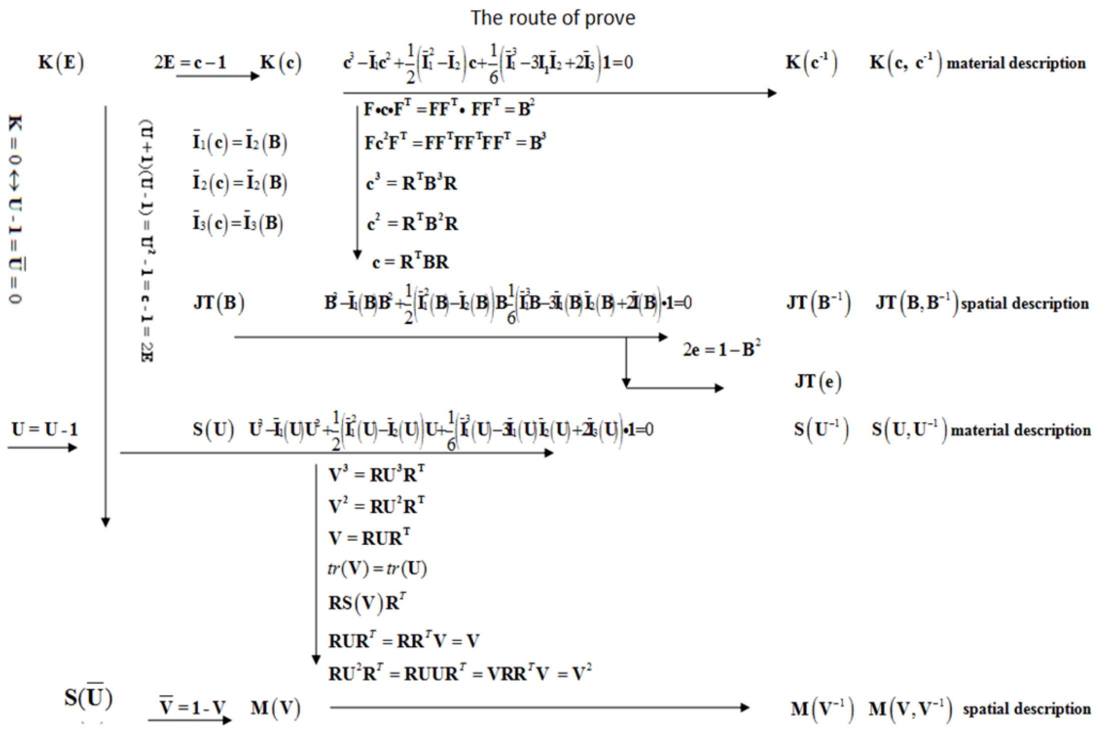

Figure 1. The route of prove

3. Non-linear Thermo-elastic Constitutive Equation of isotropic hyperelastic materials.

Wineman and Pipkin[13-14] was proved a complete representation of a tensor polynomial can be regarded as a complete representation of a general tensor function.The literature[15] was in the study of

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the nonlinear constitutive equation of hyperelastic materials,the structural proof method was adopted,and give the non-linear constitutive equation of isotropic elastomer,And it is pointed out that its constitutive equation is complete and irreducible.Considering the influence of temperature and strain bivariate on the non-linear hyperelastic isotropic materials,we can prove the constitutive equation is expressed in a complete and irreducible quad-degree polynomial form with $\bar{I}_1, \bar{I}_2, \bar{I}_3$ as the 1st, 2nd and 3rd main invariant of $\boldsymbol{E}$ and temperature $\boldsymbol{T}$,as:

$$
\begin{aligned}
\boldsymbol{K} = [& k_1\bar{I}_1 + k_3\bar{I}_1^2 + k_4\bar{I}_2 + k_6\bar{I}_1^3 + 2k_7\bar{I}_1\bar{I}_2 + k_8\bar{I}_3 + k_{10}\bar{I}_1^4 + 3k_{11}\bar{I}_1^2\bar{I}_2 + k_{12}\bar{I}_2^2 \\
& + 2k_{13}\bar{I}_1\bar{I}_3 + (\delta_1 + \delta_3\bar{I}_1 + \delta_7\bar{I}_1^2 + \delta_8\bar{I}_2 + \delta_{16}\bar{I}_1^3 + \delta_{17}\bar{I}_3)T + (\delta_2 + \delta_6\bar{I}_1 + \delta_{13}\bar{I}_1^2 \\
& + \delta_{14}\bar{I}_2)T^2 + (\delta_5 + \delta_{15}\bar{I}_1)T^3 + \delta_{12}T^4]\boldsymbol{1} + 2[k_2 + k_4\bar{I}_1 + k_7\bar{I}_1^2 + k_9\bar{I}_2 + k_{11}\bar{I}_1^3 \\
& + 2k_{12}\bar{I}_1\bar{I}_2 + k_{14}\bar{I}_3 + (\delta_4 + \delta_{10}\bar{I}_1 + \delta_{20}\bar{I}_1^2 + \delta_{21}\bar{I}_2)T + (\delta_9 + \delta_{19}\bar{I}_1)T^2 + \delta_{18}T^3]\boldsymbol{E} \\
& + 3[k_5 + k_8\bar{I}_1 + k_{13}\bar{I}_1^2 + k_{14}\bar{I}_2 + (\delta_{11} + \delta_{23}\bar{I}_1)T + \delta_{22}T^2]\boldsymbol{E}^2
\end{aligned}
\qquad (6)
$$

$\boldsymbol{1}$ is unit tensor， K is second kind of Piola-kirchhoff stress tensor， $\boldsymbol{E}$ is Lagrange strain tensor. $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ as the1st, 2nd and 3rd main invariant of $\boldsymbol{E}$. $k_1, k_2, k_3..., k_{14}$ is 14 independent elastic constants, $\delta_1, \delta_2, \delta_3..., \delta_{23}$ are 23 coefficients of temperature.

4. Example in Application.

For isotropic Green elastomer materials,The stress tensor $\boldsymbol{K}$ and strain energy function $\boldsymbol{W}$ have the following integral relation.

$$
W(\mathbf{E}) = \int \mathbf{K} : d\mathbf{c} \qquad (7)
$$

Take Eq. (5) into Eq. (7),Consider non-compressive material:

$$
du = |\mathbf{F}|\, dv = \mathrm{J}\, dv \,,\; J_3 = \det c = |F'F| = J^2 \text{ that is: } \mathrm{J} = 1 \quad \mathrm{J}_3 = 1
$$

The strain energy function of an incompressible object can be introduced:

$$
\begin{aligned}
W = {} & \delta_{11}(l_1 - 3)^5 + (\delta_7 + \delta_{15}T)(l_1 - 3)^4 + (\delta_4 + \delta_{16}T + \delta_{25}T^2)(l_1 - 3)^3 + (\delta_2 \\
& + \delta_{17}T + \delta_{26}T^2 + \delta_{32}T^3)(l_1 - 3)^2 + (\delta_1 + \delta_{18}T + \delta_{28}T^2 + \delta_{31}T^3 + \delta_{34}T^4)(l_1 - 3) \\
& + (\delta_9 + \delta_{22}T)(l_2 - 3)^2 + (\delta_3 - \delta_{23}T - \delta_{29}T^2 - \delta_{33}T^3)(l_2 - 3) + (\delta_5 - \delta_{20}T \\
& - \delta_{27}T^2)(l_1 - 3)(l_2 - 3) + (\delta_8 - \delta_{19}T)(l_1 - 3)^2(l_2 - 3) + \delta_{12}(l_1 - 3)^3(l_2 - 3) \\
& + \delta_{13}(l_1 - 3)(l_2 - 3)^2
\end{aligned}
\qquad (8)
$$

The experimental data from L.R.G which explain the relationship between the nominal stress and elongation ratio of vulcanized natural rubber at different temperatures is used by the MATLAB with the method of multi-objective regression fitting is compared with fitting curve.As shown in the following figures:

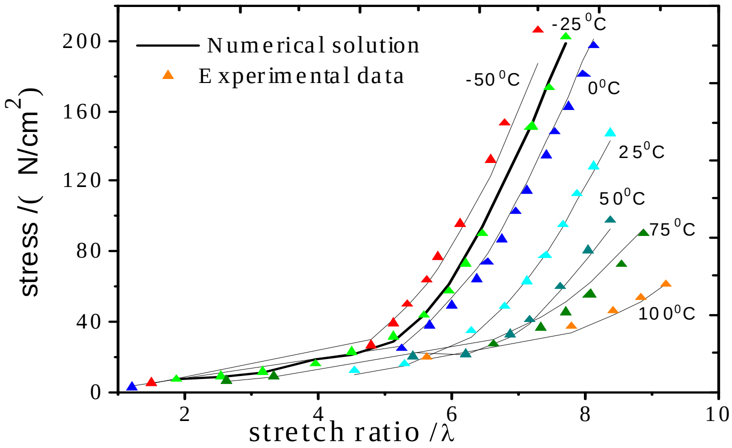

Figure 2. Stress-Stretch ratio cure at different temperature

Table 1. Rubber elasticity coefficient obtained by regression analysis

δ_1	4.22576e3	δ_2	0.64676e3	δ_3	-4.72051e3
δ_4	-0.01056	δ_5	0	δ_7	-1.69222e-2
δ_8	0.06212	δ_9	-2.70215	δ_{11}	6.570e-06
δ_{12}	0.00032	δ_{13}	-0.33660	δ_{15}	-5.02533e-06
δ_{16}	-4.26388e-3	δ_{17}	-0.00076	δ_{18}	-0.04225
δ_{19}	-4.58871e-2	δ_{20}	-0.00231	δ_{22}	0.00892
δ_{23}	-0.05857	δ_{25}	-9.2912e-07	δ_{26}	4.11422e-4
δ_{27}	2.86147e-3	δ_{28}	2.19794e-2	δ_{29}	2.55284e-2
δ_{31}	7.01505e-05	δ_{32}	-8.54e-09	δ_{33}	2.19488e-4
δ_{34}	-2.7512e-07				

Note:The δ_1 to δ_{14} is related to the mechanical properties of the material,and δ_{15} to δ_{34} is related to the temperature of the material.

It can be seen from the figure,taking the non-linear thermo-elastic constitutive equation of isotropic hyperelastic materials(Eq.(6)) into The tensile experiment was applied to different temperature(-50℃、-25℃、0℃、25℃ 50℃、75℃、100℃)of vulcanized rubber, and the results show that the fitting accuracy is satisfactory.At the same time, the elastic constants and temperature coefficients of vulcanized rubber can be determined.

5. Conclusion

1.Non-linear constitutive equation of isotropic elastomer has 16 different form,in essence there is no difference in the application of different forms of constitutive equations,but there are differences in the degree of difficulty,it can be seen that Eq(8) is deduced by Eq(5) considered the non-compressive material,using Eq(8)is easier than Eq(6).

2.Non-linear Thermo-elastic Constitutive Equation of isotropic hyperelastic materials is complete and irreducible has 14 independent elastic constants and 23 coefficients of temperature.

3.The elastic constants and temperature coefficients of specific materials can be determined according to the experimen.

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