

The Construction of Iterative Solutions to A kind of BVP

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Abstract. We focus on the positive iterative solution to the problem of a kind of BVP with the p-Laplace operator. We will study its positive solutions. The specific methods used in our work is the combination of fixed point theorem and monotone iterative technique. What's more, at the end of this work, an example is presented to show that this method can be well used to get the main results.

1. Introduction

Inspired by some works of literature, we focus on the positive iterative solution to the following BVP

$$(\phi_p(u''))'(t) = q(t)f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \quad (1)$$

$$u(0) = u'(1) = u''(1) = 0, \quad (2)$$

where $p > 1$, and

$(H) : f(t, x, y, z) \in C([0, 1] \times [0, +\infty) \times \mathbb{R} \times (+\infty, 0] \rightarrow [0, +\infty))$, $q(t)$ is a nonnegative continuous function defined on $[0, 1]$, and $q(t) \neq 0$ on any subinterval of $(0, 1)$.

When it comes to third order equations, a lot of authors do some works. For example, in [1], Li did some research work on the singular BVP

$$\begin{cases} u'''(t) = \lambda a(t)f(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u''(1) = 0, \end{cases} \quad (3)$$

They obtained their results by choosing the Krasnoselskii's theorem of cone expansion and compression type. In [2, 3], the authors considered the BVP

$$\begin{cases} u'''(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \quad (4)$$

They utilized the upper and lower solutions method. B. Hopkins and N. kosmatov in [4], obtain the existence of results to

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad a.e. 0 < t < 1, \quad (5)$$

satisfying

$$\begin{aligned} u(0) &= u'(0) = u''(1) = 0, \\ \text{and } u(0) &= u'(1) = u''(1) = 0, \end{aligned} \quad (6)$$

Their choice is the Leray-Schauder continuation principle.

Inspired by them, we focus on the BVP (1)(2). We will construct iterative sequence to approximate the solutions.

2. Preliminaries

Consider $E = C^2[0, 1]$, define the norm

$$\|u\| := \max \{ \max_{0 \leq t \leq 1} |u(t)|, \max_{0 \leq t \leq 1} |u'(t)|, \max_{0 \leq t \leq 1} |u''(t)| \}.$$

And let $E_+ = C_+[0,1] = \{u \in E \mid u(t) \geq 0, t \in [0,1]\}$,

the cone $P \subset E$ by $P = \{u \in E \mid u(t) \geq 0, u \text{ is concave and nondecreasing}\}$.

During our process, the condition (H) is always true.

Lemma 1. Assume $y \in C^2[0,1]$ with $(\phi_p(y''(t)))' \in L^1[0,1]$ such that

$$(\phi_p(y''(t)))' \geq 0, \quad 0 \leq t \leq 1,$$

$$y(0) = y'(1) = y''(1) = 0.$$

we have that $y \in P$.

For $\forall x \in C_+[0,1]$, suppose u is a solution of the problem (1)(2), then

$$u''(t) = -\phi_p^{-1} \left(\int_t^1 q(s) f(t, x(s), x'(s), x''(s)) ds \right), \quad u'(t) = \int_t^1 \phi_p^{-1} \left(\int_s^1 q(r) f(r, x(r), x'(r), x''(r)) dr \right) ds$$

and

$$u(t) = \int_0^t \left[\int_s^1 \phi_p^{-1} \left(\int_\tau^1 q(r) f(r, x(r), x'(r), x''(r)) dr \right) d\tau \right] ds, \quad 0 \leq t \leq 1.$$

3. Main Results

Theorem 1. Assume that (H) hold, and $\exists a > 0$, then

(S1): $f(t, x_1, y_1, z_1) \leq f(t, x_2, y_2, z_2)$, for any $0 \leq t \leq 1, 0 \leq x_1 \leq x_2 \leq a, 0 \leq y_1 \leq y_2 \leq a, -a \leq z_1 \leq z_2 \leq 0$;

(S2): $\max_{0 \leq t \leq 1} f(t, a, a, -a) \leq \phi_p \left(\frac{a}{A} \right)$, where $A = \phi_p^{-1} \left(\int_0^1 q(s) ds \right)$;

(S3): $f(t, 0, 0, 0) \neq 0$, for $0 \leq t \leq 1$;

Thus, we can say that the BVP (1)(2) has two positive concave nondecreasing solutions w^* and v^* , then

$$0 < w^* \leq a, \quad 0 \leq (w^*)' \leq a, \quad -a \leq (w^*)'' \leq 0,$$

$$\text{and } \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} T^n w_0 = w^*, \quad \lim_{n \rightarrow \infty} (w_n)' = \lim_{n \rightarrow \infty} (T^n w_0)' = (w^*)',$$

$$\lim_{n \rightarrow \infty} (w_n)'' = \lim_{n \rightarrow \infty} (T^n w_0)'' = (w^*)'',$$

$$\text{where } w_0(t) = at(1 - \frac{t}{2}), \quad 0 \leq t \leq 1,$$

and

$$0 < v^* \leq a, \quad 0 \leq (v^*)' \leq a, \quad -a \leq (v^*)'' \leq 0,$$

$$\text{and } \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} T^n v_0 = v^*, \quad \lim_{n \rightarrow \infty} (v_n)' = \lim_{n \rightarrow \infty} (T^n v_0)' = (v^*)',$$

$$\lim_{n \rightarrow \infty} (v_n)'' = \lim_{n \rightarrow \infty} (T^n v_0)'' = (v^*)'',$$

$$\text{where } v_0(t) = 0, \quad 0 \leq t \leq 1,$$

where

$$(Tu)(t) = \int_0^t \left[\int_s^1 \phi_p^{-1} \left(\int_\tau^1 q(r) f(r, x(r), x'(r), x''(r)) dr \right) d\tau \right] ds, \quad 0 \leq t \leq 1. \quad (7)$$

There are two iterative schemes. One is $w_0(t) = at(1 - \frac{t}{2})$, $w_{n+1} = Tw_n = T^n w_0$, $n = 0, 1, 2, \dots$ and the other is

$$v_0(t) = 0, v_{n+1} = Tv_n = T^n v_0, n = 0, 1, 2, \dots$$

Proof. Our proof is divided into the following steps.

Step 1. Let us define $T: P \rightarrow E$ by (7), now, let us verify that $T: P \rightarrow E$ is completely continuous.

We can see that $\forall u \in P$, there is $Tu \in C^2[0,1]$. then

$$(Tu)''(t) = -\phi_p^{-1} \left(\int_t^1 q(s) f(s, u(s), u'(s), u''(s)) ds \right), \quad (8)$$

$$(Tu)'(t) = \int_t^1 \phi_p^{-1} \left(\int_s^1 q(r) f(r, u(r), u'(r), u''(r)) dr \right) ds. \quad (9)$$

What's more, $(Tu)''(t) \leq 0$, from which it can be easily derived that Tu is concave. And we can also prove that $(Tu)(t)$ is nondecreasing. Then, $(Tu)(t) \geq (Tu)(0) = 0, 0 \leq t \leq 1$. Which means that $T: P \rightarrow P$. According to the conventional proof method, it turns out that $T: P \rightarrow P$ is an operator which is completely continuous.

Step 2. Let $\bar{P}_a = \{u \in P \mid \|u\| \leq a\}$. In this step, let us prove that $T: \bar{P}_a \rightarrow \bar{P}_a$.

When $u \in \overline{P_a}$, then $\|u\| \leq a$, so

$$\begin{aligned} 0 &\leq u(t) \leq u(1) = \max_{0 \leq t \leq 1} |u(t)| \leq \|u\| \leq a, \\ 0 &= u'(1) \leq u'(t) \leq \max_{0 \leq t \leq 1} |u'(t)| \leq \|u'\| \leq a, \\ -a &\leq -\|u\| \leq -\max_{0 \leq t \leq 1} |u''(t)| = u''(0) \leq u''(t) \leq 0. \end{aligned}$$

By (S1), (S2), then

$$0 \leq f(t, u(t), u'(t), u''(t)) \leq f(t, a, a, -a) \leq \max_{0 \leq t \leq 1} f(t, a, a, -a) \leq \phi_p\left(\frac{a}{A}\right), \quad \text{for } 0 \leq t \leq 1.$$

Combing with (7), (8), (9), we obtain

$$\begin{aligned} (Tu)(1) &= \int_0^1 \left[\int_s^1 \phi_p^{-1} \left(\int_\tau^1 q(r) f(r, x(r), x'(r), x''(r)) d\tau \right) d\tau \right] ds \\ &\leq \frac{a}{A} \phi_p^{-1} \left(\int_0^1 q(s) ds \right) = a, \\ (Tu)'(0) &= \int_0^1 \phi_p^{-1} \left(\int_s^1 q(r) f(r, u(r), u'(r), u''(r)) dr \right) ds \\ &\leq \frac{a}{A} \phi_p^{-1} \left(\int_0^1 q(s) ds \right) = a, \end{aligned}$$

and

$$\begin{aligned} -(Tu)''(0) &= \phi_p^{-1} \left(\int_0^1 q(s) f(s, u(s), u'(s), u''(s)) ds \right) \\ &\leq \frac{a}{A} \phi_p^{-1} \left(\int_0^1 q(s) ds \right) = a. \end{aligned}$$

Thus

$$\begin{aligned} \|Tu\| &= \max \{ \max_{0 \leq t \leq 1} |(Tu)(t)|, \max_{0 \leq t \leq 1} |(Tu)'(t)|, \max_{0 \leq t \leq 1} |(Tu)''(t)| \} \\ &= \max \{ (Tu)(1), (Tu)'(0), -(Tu)''(0) \}. \end{aligned}$$

Hence, from the above discussion, we can get $T: \overline{P_a} \rightarrow \overline{P_a}$.

Step 3. We will construct iterative sequences to approximate the solutions.

Firstly, let us verify the monotone property of T with respect to u . By (S1) we have, for any $u_i \in P$ ($i=1,2$) with $u_1 \leq u_2$, $u'_1 \leq u'_2$ and $u''_1 \geq u''_2$. Then from the definition of T , we can easily get $Tu_1 \leq Tu_2$.

Set $w_0(t) = at(1 - \frac{t}{2})$, $0 \leq t \leq 1$, then $w_0(t) \in \overline{P_a}$. Set $w_1 = Tw_0$, then $w_1 \in \overline{P_a}$. At this point, let us construct an iterative sequence $w_{n+1} = Tw_n = T^n w_0$, $n=1,2,\dots$. It is known that $T: \overline{P_a} \rightarrow \overline{P_a}$, so we have that $w_n \in T\overline{P_a} \subseteq \overline{P_a}$, $n=1,2,\dots$. Thus, we can judge that the set $\{w_n\}_{n=1}^\infty$ is sequential and compact.

Combing with (7), (8), (9), we have

$$\begin{aligned} w_1(t) &= Tw_0(t) \\ &= \int_0^t \left[\int_s^1 \phi_p^{-1} \left(\int_\tau^1 q(r) f(r, w_0(r), w'_0(r), w''_0(r)) d\tau \right) d\tau \right] ds \\ &\leq \frac{a}{A} \phi_p^{-1} \left(\int_0^1 q(s) ds \right) \int_0^t \left[\int_s^1 d\tau \right] ds \\ &\leq at(1 - \frac{t}{2}) = w_0(t), \quad 0 \leq t \leq 1, \\ w'_1(t) &= (Tw_0)'(t) \\ &= \int_t^1 \phi_p^{-1} \left(\int_s^1 q(r) f(r, w_0(r), w'_0(r), w''_0(r)) dr \right) ds \\ &\leq \frac{a}{A} \phi_p^{-1} \left(\int_0^1 q(s) ds \right) (1-t) = a(1-t) = w'_0(t), \quad 0 \leq t \leq 1, \end{aligned}$$

and

$$\begin{aligned} w''_1(t) &= (Tw_0)''(t) \\ &= -\phi_p^{-1} \left(\int_t^1 q(s) f(s, w_0(s), w'_0(s), w''_0(s)) ds \right) \\ &\geq -\frac{a}{A} \phi_p^{-1} \left(\int_0^1 q(s) ds \right) = -a = w''_0(t), \quad 0 \leq t \leq 1. \end{aligned}$$

Since the iterative sequence, it's natural that we can get it

$$w_2(t) = Tw_1(t) \leq Tw_0(t) = w_1(t), \quad 0 \leq t \leq 1, \quad w_2'(t) = (Tw_1)'(t) \leq (Tw_0)'(t) = w_1'(t), \quad 0 \leq t \leq 1,$$

$$w_2''(t) = (Tw_1)''(t) \geq (Tw_0)''(t) = w_1''(t), \quad 0 \leq t \leq 1,$$

After calculation, then

$$w_{n+1} \leq w_n, \quad w_{n+1}'(t) \leq w_n'(t), \quad w_{n+1}''(t) \geq w_n''(t), \quad 0 \leq t \leq 1, \quad n=0,1,2,\dots$$

Thus, we can say that $\exists w^* \in \overline{P}_a$ s.t. $w_n \rightarrow w^*$. Because of the continuity of T and $w_{n+1} = Tw_n$, we get $Tw^* = w^*$.

Based on the above conclusion. Set $v_0(t) = 0, 0 \leq t \leq 1$, then $v_0(t) \in \overline{P}_a$. Let $v_1 = Tv_0 = T0$, then $v_1 \in \overline{P}_a$. Similarly, here we construct an iterative sequence again. Denote $v_{n+1} = Tv_n = T^n v_0, n=0,1,2,\dots$. Due to $T: \overline{P}_a \rightarrow \overline{P}_a$, we have $v_n \in T\overline{P}_a \subseteq \overline{P}_a, n=1,2,\dots$

Because $v_1 = Tv_0 = T0 \in \overline{P}_a$, we have

$$v_1(t) = Tv_0(t) = (T0)(t) \geq 0, \quad 0 \leq t \leq 1, \quad v_1'(t) = Tv_0'(t) = (T0)'(t) \geq 0, \quad 0 \leq t \leq 1,$$

$$v_1''(t) = (Tv_0)''(t) = (T0)''(t) \leq 0, \quad 0 \leq t \leq 1.$$

Besides

$$v_2(t) = Tv_1(t) \geq (T0)(t) = v_1(t), \quad 0 \leq t \leq 1, \quad v_2'(t) = Tv_1'(t) \geq (T0)'(t) = v_1'(t), \quad 0 \leq t \leq 1,$$

$$v_2''(t) = (Tv_1)''(t) \leq (T0)''(t) = v_1''(t), \quad 0 \leq t \leq 1.$$

Through a similar calculation argument then

$$v_{n+1} \geq v_n, \quad v_{n+1}'(t) \geq v_n'(t), \quad v_{n+1}''(t) \leq v_n''(t), \quad 0 \leq t \leq 1, \quad n=0,1,2,\dots$$

Hence, $\exists v^* \in \overline{P}_a$ s.t. $v_n \rightarrow v^*$. Because the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

Form previous discussions, we have proved w^* and v^* are two solutions to (1)(2).

The theorem is proved here.

Corollary 1. Hypothesize that $(H), (S1), (S3)$ hold, and $\exists a > 0$, s.t.

$$(C1): \lim_{\ell \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, \ell, a, -a)}{\ell^{p-1}} \leq \frac{1}{A^{p-1}}, \quad (\text{particularly, } \lim_{\ell \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, \ell, a, -a)}{\ell^{p-1}} = 0).$$

Then the problem (1)(2) has two concave nondecreasing positive solutions w^* and v^* , such that the Theorem 1 is true.

Corollary 2. Hypothesize that $(H), (S3)$ hold, and $\exists 0 < a_1 < a_2 < \dots < a_n$, then

$$(C2): f(t, x_1, y_1, z_1) \leq f(t, x_2, y_2, z_2) \text{ for any } 0 \leq t \leq 1, 0 \leq x_1 \leq x_2 \leq a_k, 0 \leq y_1 \leq y_2 \leq a_k, \\ -a_k \leq z_2 \leq z_1 \leq 0, k=1,2,\dots,n;$$

$$(C3): \max_{0 \leq t \leq 1} f(t, a_k, a_k, -a_k) \leq \phi_p\left(\frac{a_k}{A}\right), \quad k=1,2,\dots,n.$$

Then the problem (1)(2) has $2n$ concave nondecreasing positive solutions w_k^* and v_k^* , such that

$$0 < w_k^* \leq a, \quad 0 \leq (w_k^*)' \leq a, \quad -a \leq (w_k^*)'' \leq 0,$$

$$\text{and } \lim_{n \rightarrow \infty} w_{k_n} = \lim_{n \rightarrow \infty} T^n w_{k_0} = w_k^*, \quad \lim_{n \rightarrow \infty} (w_{k_n})' = \lim_{n \rightarrow \infty} (T^n w_{k_0})' = (w_k^*)', \\ \lim_{n \rightarrow \infty} (w_{k_n})'' = \lim_{n \rightarrow \infty} (T^n w_{k_0})'' = (w_k^*)'',$$

$$\text{where } w_{k_0}(t) = a_k t(1 - \frac{t}{2}), \quad 0 \leq t \leq 1,$$

and

$$0 < v_k^* \leq a, \quad 0 \leq (v_k^*)' \leq a, \quad -a \leq (v_k^*)'' \leq 0,$$

$$\text{and } \lim_{n \rightarrow \infty} v_{k_n} = \lim_{n \rightarrow \infty} T^n v_{k_0} = v_k^*, \quad \lim_{n \rightarrow \infty} (v_{k_n})' = \lim_{n \rightarrow \infty} (T^n v_{k_0})' = (v_k^*)', \\ \lim_{n \rightarrow \infty} (v_{k_n})'' = \lim_{n \rightarrow \infty} (T^n v_{k_0})'' = (v_k^*)'',$$

$$\text{where } v_{k_0}(t) = 0, \quad 0 \leq t \leq 1,$$

and $(Tu)(t)$ is defined as the same as (7).

There are also some iterative schemes. $w_{k_0}(t) = a_k t(1 - \frac{t}{2}), w_{k_{n+1}} = Tw_{k_n} = T^n w_{k_0}, k=1,2,\dots,n=0,1,2,\dots$, and $v_{k_0}(t) = 0, v_{k_{n+1}} = Tv_{k_n} = T^n v_{k_0}, k=1,2,\dots,n=0,1,2,\dots$

Corollary 3. Hypothesize that $(H), (C2), (S3)$ hold, and $\exists 0 < a_1 < a_2 < \dots < a_n$, then

$$(C4): \lim_{\ell \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, \ell, a_k, -a_k)}{\ell^{p-1}} \leq \frac{1}{A^{p-1}}, \quad (\text{particularly, } \lim_{\ell \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, \ell, a_k, -a_k)}{\ell^{p-1}} = 0). \quad k = 1, 2, \dots, n$$

Then the BVP (1)(2) has $2n$ concave nondecreasing positive solutions w_k^* and v_k^* , then the conclusion of Corollary 2 is true.

4. Example

Example 1. Set $p = \frac{3}{2}$, $q(t) = 1$, let us consider the problem

$$(|u''(t)|^{\frac{1}{2}} u''(t))' = f(t, u(t), u'(t), u''(t)), \quad 0 < t < 1, \quad (10)$$

$$u(0) = u'(1) = u''(1) = 0, \quad (11)$$

Where $f(t, x, y, z) = \frac{1}{4}t^2 + \frac{1}{8}x + \frac{1}{8}y - \frac{1}{8}z$, Set $a = 4$, then

So, $f(t, x, y)$ satisfies:

(1): $f(t, x_1, y_1, z_1) \leq f(t, x_2, y_2, z_2)$ for any $0 \leq t \leq 1, 0 \leq x_1 \leq x_2 \leq 4, 0 \leq y_1 \leq y_2 \leq 4, -4 \leq z_2 \leq z_1 \leq 0$;

(2): $\max_{0 \leq t \leq 1} f(t, a, a, -a) = f(1, 4, 4, -4) < \phi_2(\frac{a}{2}) = 2$;

(3): $f(t, 0, 0) \neq 0$ for $0 \leq t \leq 1$.

then because of theorem 1, the BVP (10)(11) has two concave nondecreasing positive solutions w^* and v^* , s.t.

$$0 < w^* \leq 4, \quad 0 \leq (w^*)' \leq 4, \quad -4 \leq (w^*)'' \leq 0,$$

$$\text{and } \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} T^n w_0 = w^*, \quad \lim_{n \rightarrow \infty} (w_n)' = \lim_{n \rightarrow \infty} (T^n w_0)' = (w^*)',$$

$$\lim_{n \rightarrow \infty} (w_n)'' = \lim_{n \rightarrow \infty} (T^n w_0)'' = (w^*)'',$$

$$\text{where } w_0(t) = 4t - 2t^2, \quad 0 \leq t \leq 1,$$

and

$$0 < v^* \leq 4, \quad 0 \leq (v^*)' \leq 4, \quad -a \leq (v^*)'' \leq 0,$$

$$\text{and } \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} T^n v_0 = v^*, \quad \lim_{n \rightarrow \infty} (v_n)' = \lim_{n \rightarrow \infty} (T^n v_0)' = (v^*)',$$

$$\lim_{n \rightarrow \infty} (v_n)'' = \lim_{n \rightarrow \infty} (T^n v_0)'' = (v^*)'',$$

$$\text{where } v_0(t) = 0, \quad 0 \leq t \leq 1,$$

For $n = 0, 1, 2, \dots$ The two iterative schemes are

$$w_0(t) = 4t - 2t^2, \quad 0 \leq t \leq 1,$$

$$w_1(t) = (Tw_0)(t) = -\frac{1}{12}t^4 + \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{3}t, \quad 0 \leq t \leq 1,$$

$$\dots \dots \dots$$

$$w_{n+1}(t) = \int_0^t \left[\int_s^1 \phi_p^{-1} \left(\int_r^1 \frac{1}{4}r^2 + \frac{1}{8}w_n(r) + \frac{1}{8}w_n'(r) - \frac{1}{8}w_n''(r) dr \right) d\tau \right] ds, \quad 0 \leq t \leq 1.$$

$$v_0(t) = 0,$$

$$v_1(t) = (Tv_0)(t) = -\frac{1}{8064}t^8 + \frac{1}{1440}t^5 - \frac{1}{288}t^2 + \frac{1}{224}t, \quad 0 \leq t \leq 1,$$

$$\dots \dots \dots$$

$$v_{n+1}(t) = \int_0^t \left[\int_s^1 \phi_p^{-1} \left(\int_r^1 \frac{1}{4}r^2 + \frac{1}{8}v_n(r) + \frac{1}{8}v_n'(r) - \frac{1}{8}v_n''(r) dr \right) d\tau \right] ds, \quad 0 \leq t \leq 1.$$

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