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Mathematical Model to estimate the wind power using four-parameter Burr distribution

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Abstract: When the real probability of wind speed in the same position needs to be described, the four-parameter Burr distribution is more suitable than other distributions. This paper introduces its important properties and characteristics. Also, the application of the four-parameter Burr distribution in wind speed prediction is discussed, and the expression of probability distribution of output power of wind turbine is deduced.

1. Introduction

Driven by environmental concerns, energy independence, and limitation of fossil fuel resource, wind power is becoming a preferred form of energy in the world [1] [2]. Especially, China's wind industry is gaining momentum. The installed capacity of wind power increases year by year.

However, due to the intermittent and uncontrollability of wind power, large-scale deployment can make an unpredictable effect on electric system. So, existing system analysis methods need to be improved.

Probabilistic assessment acts a pivotal part in analysis, particularly production cost and reliability analysis. Probabilistic wind power generation model is an effective method that integrates wind power into existing analysis architecture. Characteristics of wind energy systems are mainly determined by wind velocity distribution [3]. Probability density function (PDF) of wind speed is of great significance for many projects, such as selection of wind farms [4] [5]. Morgan proved that the probability distribution is of great significance in estimating energy production for wind turbine design [6].

At present, through arithmetic average of the wind, most wind energy investigations are obtained. Some researchers estimate wind energy, which is based on elaborated wind-speed statistics. And some researches study application of four-parameter Burr distribution in wind velocity. The generalized gamma, Weibull distributions and four-parameter Burr distribution are compared, and the final consequence is that four- parameter Burr distribution is more suitable[7].In this paper, we firstly suppose that four-parameter Burr distribution is the most suitable and exact fit for the empirical distribution of the wind speed measurement samples. This paper is intended as an introduction to the use of the Four-parameter Burr distribution to model single-site hourly average wind speeds (i.e., wind speeds at a given wind farm). We derive a probability distribution for the electric power output of a wind turbine with given cut-in, cut-out and rated wind speeds. Our focus will be abstract and mathematical rather than concrete and empirical and running some algorithms and statistical tests with

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standard software (e.g. Mat lab).

After this introduction, the structure of the paper is as follows: In Section 2, some properties of four parameter Burr distribution for the wind speed are given. In Section 3, the probability distribution of the wind power is deduced from the probability distribution of the wind speed. Then the mean of the wind power is given. In Section 4, Turbine site matching is discussed. The conclusion is presented in Section 5.

2. Burr Distribution

The Burr distribution has a various form for different case. It is often compared with normal distribution and then applied to statistical modeling. Burr expounded the Burr family distribution in [8]. Among Burr family distributions, Tadikamalla [9] showed Burr distribution has four parameters (1ocation, scale, and two shape parameters cover wider distributions of different shapes). Burr distribution could approximate familiar distributions, appropriately, such as Normal, and so on. Especially, scale and location parameters make it have a preferable result of fitting the wind speed data and the final curve belongs to a slight positive skewness. [10]. Valerio' final analysis shows that the model has a preferable accuracy for fitting empirical data [11].

The PDF of a random variable wind speed V following a Burr distribution with four parameters is:

$$f_V(x)=\begin{cases}\dfrac{\alpha k}{\beta}\left(\dfrac{x-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\dfrac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1}, & \gamma\le x<+\infty\\ 0, & -\infty<x<\gamma\end{cases} \tag{1}$$

Where, $\beta>0$ represents the Scale Parameter; $k>0$, $\alpha>0$ represent the Dimensionless Shape Parameters; and $\gamma>0$ represents the Location Parameter [12]. A random variable wind speed V following a Burr distribution with four parameters is denoted by

$$V:\ Burr(\alpha,\beta,\gamma,k).$$

***Theorem* 1** Let a random variable wind speed V follow a Burr distribution with four parameters. Then the Cumulative Distribution Function (CDF) of V is expressed as follows

$$F_V(x)=1-\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k} \tag{2}$$

Proof: By the fundamental theorem of calculus, we have that

$$F_V(x)=\int_{-\infty}^{x}f_V(t)dt$$

$$=\int_{\gamma}^{x}\frac{\alpha k}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{t-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1}dt$$

$$=1-\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

***Theorem* 2** Let a wind speed V follow a Burr distribution with four parameters, then the mean of V is $E_{(V)}=k\beta B\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+\gamma$.

Proof: The mean of V is given

$$E(V)=\int_{-\infty}^{+\infty} t f_V(t)dt=\int_{\gamma}^{+\infty} t\frac{\alpha k}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{t-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1}dt.$$

Let $t-\gamma=u$ and $z=\dfrac{1}{1+\left(\frac{u}{\beta}\right)^{\alpha}}$, then

$$E(V)=\int_{0}^{+\infty}(u+\gamma)\frac{\alpha k}{\beta}\left(\frac{u}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{u}{\beta}\right)^{\alpha}\right)^{-k-1}du$$

$$=\frac{\alpha k}{\beta}\int_{0}^{+\infty}u\left(\frac{u}{\beta}\right)^{\alpha-1}\left[1+\left(\frac{u}{\beta}\right)^{\alpha}\right]^{-k-1}du+\gamma=k\int_{0}^{1}\beta(1-z)^{\frac{1}{\alpha}}z^{-\frac{1}{\alpha}}z^{k-1}dz+\gamma$$

$$=k\beta\int_{0}^{1}(1-z)^{\frac{1}{\alpha}}z^{k-1-\frac{1}{\alpha}}dz+\gamma=k\beta B\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+\gamma.$$

***Theorem* 3** Let the wind speed V following a Burr distribution with four parameters, then two order moment of V is given as

$$E\left(V^2\right)=k\beta^2B\left(k-\frac{2}{\alpha},\frac{2}{\alpha}+1\right)+2\gamma k\beta B\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+3\gamma^2.$$

Proof: Two order moment of V is following as

$$E\left(V^2\right)=\int_{-\infty}^{+\infty} t^2 f_V(t)dt=\int_{\gamma}^{+\infty} t^2\frac{\alpha k}{\beta}\left(\frac{t-\gamma}{\beta}\right)^{\alpha-1}\left[1+\left(\frac{t-\gamma}{\beta}\right)^{\alpha}\right]^{-k-1}dt.$$

Let $t-\gamma=u$ and $z=\dfrac{1}{1+\left(\frac{u}{\beta}\right)^{\alpha}}$, then

$$E\left(V^2\right)=\int_{0}^{+\infty}(u+\gamma)^2\frac{\alpha k}{\beta}\left(\frac{u}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{u}{\beta}\right)^{\alpha}\right)^{-k-1}du$$

$$=\frac{\alpha k}{\beta}\int_{0}^{+\infty}u^2\left(\frac{u}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{u}{\beta}\right)^{\alpha}\right)^{-k-1}du+2\gamma E(V)+\gamma^2$$

$$=k\int_{0}^{1}\beta^2z^{k-1}(1-z)^{\frac{2}{\alpha}}z^{-\frac{2}{\alpha}}dx+2\gamma E(V)+\gamma^2$$

$$=k\beta^2\int_{0}^{1}z^{k-1-\frac{2}{\alpha}}(1-z)^{\frac{2}{\alpha}}dx+2\gamma E(V)+\gamma^2$$

$$=k\beta^2B\left(k-\frac{2}{\alpha},\frac{2}{\alpha}+1\right)+2\gamma k\beta B\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+3\gamma^2.$$

***Theorem* 4** Let the wind speed V follow a Burr distribution with four parameters, then the variance of V is given as

$$D(V)=k\beta^2B\left(k-\frac{2}{\alpha},\frac{2}{\alpha}+1\right)-k^2\beta^2B^2\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+2\gamma^2.$$

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Proof: According to the formula $D(V)=E(V^2)-[E(V)]^2$, we have

$$D(V)=E(V^2)-[E(V)]^2$$

$$=k\beta^2 B\left(k-\frac{2}{\alpha},\frac{2}{\alpha}+1\right)+2\gamma k\beta B\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+3\gamma^2-\left(k\beta B\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+\gamma\right)^2$$

$$=k\beta^2 B\left(k-\frac{2}{\alpha},\frac{2}{\alpha}+1\right)-k^2\beta^2 B^2\left(k-\frac{1}{\alpha},\frac{1}{\alpha}+1\right)+2\gamma^2$$

***Theorem* 5** Let $X_1, X_2, \cdots, X_n$ be n independently continuous random variables from the Burr distribution with four parameters. Then

$$P\left(\min(X_1,X_2,\cdots,X_n)>x\right)=\prod_{i=1}^{n}\left(1+\left(\frac{x-\gamma_i}{\beta_i}\right)^{\alpha_i}\right)^{k_i}.$$

If $\beta_1=\beta_2=\cdots=\beta_n=\beta,\quad \alpha_1=\alpha_2=\cdots=\alpha_n=\alpha,\quad \gamma_1=\gamma_2=\cdots=\gamma_n=\gamma,$ then

$$\min(X_1,X_2,\cdots,X_n)\sim Burr(\alpha,\beta,\gamma,k_1+k_2+\cdots+k_n).$$

Proof: It is obvious that $\min(X_1, X_2, \cdots, X_n)>x$ is equivalent to the following

$$X_i>x,\quad i=1,2,\cdots,n.$$

Since $X_1, X_2, \cdots, X_n$ are independent, we get that

$$P\left(\min(X_1,X_2,\cdots,X_n)>x\right)=\prod_{i=1}^{n}P(X_i>x)==\prod_{i=1}^{n}\left(1+\left(\frac{x-\gamma_i}{\beta_i}\right)^{\alpha_i}\right)^{k_i}.$$

The proof of the first assertion is completed. Now suppose that

$$\beta_1=\beta_2=\cdots=\beta_n=\beta,\quad \alpha_1=\alpha_2=\cdots=\alpha_n=\alpha,\quad \gamma_1=\gamma_2=\cdots=\gamma_n=\gamma.$$

Then

$$P\left(\min(X_1,X_2,\cdots,X_n)>x\right)=\prod_{i=1}^{n}\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k_i}=\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{\sum_{i=1}^{n}k_i}$$

Therefore, we have that

$$P\left(\min(X_1,X_2,\cdots,X_n)\le x\right)=1-P\left(\min(X_1,X_2,\cdots,X_n)>x\right)$$

$$=1-\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{\sum_{i=1}^{n}k_i}.$$

Hence $\min(X_1, X_2, \cdots, X_n) \sim Burr(\alpha,\beta,\gamma,k_1+k_2+\cdots+k_n)$. The proof is complete.

***Theorem* 6** The inverse Burr cumulative function can be expressed in closed form.

Proof: The cumulative distribution of Burr is that

$$F_V(x)=1-\left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

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Let $F_V(x) = y$, where $0 < y < 1$, we get that $1 - \left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k} = y$. Therefore, we have that

$$F_V^{-1}(x) = \beta\left((1-y)^{-\frac{1}{k}} - 1\right)^{\frac{1}{\alpha}} + \gamma.$$

The proof is complete.

***Theorem* 7** Let $\mu > 0$ and V : $Burr(\alpha, \beta, \gamma, k)$. Let $Y = \mu V$, then V : $Burr(\alpha, \beta, \gamma, k)$.

Proof: By Theorem 1, we get that

$$F_Y(x) = P(Y \le x) = P\left(V \le \frac{x}{\mu}\right) = F_V\left(\frac{x}{\mu}\right) = 1 - \left(1 + \left(\frac{x - \mu\gamma}{\mu\beta}\right)^{\alpha}\right)^{-k}$$

Which shows Y : $Burr(\alpha, \mu\beta, \mu\gamma, k)$.The proof is complete.

The quantified amount of four parameters in Four-parameter Burr distribution would be computed by mixed variable neighborhood search [12]. Because estimation of the parameters is not our focus, readers might see [12].

3. Wind power distribution

Most probabilistic models of wind generation based on the wind speed distribution. Several distributions, other than Gaussian distribution, have been proposed in [13].

In this section, we introduce the aerodynamic at a very elementary and simplified level of wind turbines. As we can obtain from aerodynamic principle that the power in airflow is proportional to directly proportional to wind speed cubed:

$$P_{air} = 0.5\rho A v^3.$$

Where ρ represents the Air Density; A represents the Swept Area. In fact, if output power P_{vt} of wind turbine is divided by P_{air}, we use power coefficient "C_P" to represent the ratio:

$$C_P = \frac{P_{vt}}{P_{air}}.$$

According to Betz limit, the value of C_P has a maximum $C_{P,\max} = 0.593$. We call $C_{P,\max}$ the power coefficient of the Betz turbine. Thus, the amount of energy the Betz turbine extracts is $\frac{1}{2}C_{P,\max}\rho A v^3$.

Modern wind turbines come reasonably close to the Betz limit with power coefficients upwards of $C_P \approx 0.5$ [14]. For the remainder of this paper, we will assume the electric power, generated by wind turbine, which is computed by $\frac{1}{2}C_P\rho\eta A v^3$.Where C_P is the power coefficient;

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$\eta \in (0,1)$ is efficiency constant.

In the following, we suppose $V: \ Burr(\alpha, \beta, \gamma, k)$, $P_{air} = \frac{1}{2}\rho A v^3$.A wind turbine does not operate at all wind speeds v [15]. The implied wind speed of wind turbine with power coefficient C_P and efficiency coefficient η producing power P_t is given by

$$P_t = \begin{cases} 0, & v < v_{cut-in} \\ \frac{1}{2} C_P \rho \eta A v^3, & v_{cut-in} \le v \le v_{rated} \\ \frac{1}{2} C_P \rho \eta A v_{rated}{}^3, & v_{rated} < v < v_{cut-off} \\ 0, & v \ge v_{cut-off} \end{cases}$$

Where, $v_{cut-in} < v_{rated} < v_{cut-off}$ are specified by the manufacturer.

***Theorem* 8** The power P_{air} in airflow has cumulative distribution function

$$F_{P_{air}} = 1 - \left(1 + \left(\frac{(2x)^{\frac{1}{3}} - \gamma(\rho A)^{\frac{1}{3}}}{\beta(\rho A)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k}.$$

And the density function can be written as follow:

$$f_{P_{air}}(x) = \frac{2^{\frac{1}{3}} k\alpha}{3\beta(\rho A)^{\frac{1}{3}} x^{\frac{2}{3}}} \left(\frac{(2x)^{\frac{1}{3}} - \gamma(\rho A)^{\frac{1}{3}}}{\beta(\rho A)^{\frac{1}{3}}}\right)^{\alpha-1} \left(1 + \left(\frac{(2x)^{\frac{1}{3}} - \gamma(\rho A)^{\frac{1}{3}}}{\beta(\rho A)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k-1}.$$

Proof: By the definition of cumulative distribution function and Theorem 1, we get that

$$F_{P_{air}}(x) = P(P_{air} \le x) = P\left(\frac{1}{2}\rho A v^3 \le x\right) = P\left(v \le \left(\frac{2x}{\rho A}\right)^{\frac{1}{3}}\right) = F_V\left(v \le \left(\frac{2x}{\rho A}\right)^{\frac{1}{3}}\right)$$

$$= 1 - \left(1 + \left(\frac{\left(\frac{2x}{\rho A}\right)^{\frac{1}{3}} - \gamma}{\beta}\right)^{\alpha}\right)^{-k} = 1 - \left(1 + \left(\frac{(2x)^{\frac{1}{3}} - \gamma(\rho A)^{\frac{1}{3}}}{\beta(\rho A)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k}.$$

Differentiating both sides with respect to x, we obtain that P_{air} has probability density function $f_{P_{air}}(x)$ given by

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$$f_{P_{air}}(x)=\frac{2^{\frac{1}{3}}k\alpha}{3\beta(\rho A)^{\frac{1}{3}}x^{\frac{2}{3}}}\left(\frac{(2x)^{\frac{1}{3}}-\gamma(\rho A)^{\frac{1}{3}}}{\beta(\rho A)^{\frac{1}{3}}}\right)^{\alpha-1}\left(1+\left(\frac{(2x)^{\frac{1}{3}}-\gamma(\rho A)^{\frac{1}{3}}}{\beta(\rho A)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k-1}.$$

***Theorem* 9** Let the implied wind speed of wind turbine with power coefficient C_P and efficiency coefficient η, produce power P_t, then the cumulative distribution function of P_t is got by $F_{P_t}(w)$:

If $w<0$, then $F_{P_t}(w)=0$;

If $0\le w<\frac{1}{2}C_P\rho\eta Av_{cut-in}^3$, then

$$F_{P_t}(w)=1-\left(1+\left(\frac{v_{cut-in}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}+\left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

If $\frac{1}{2}C_P\rho\eta Av_{cut-in}^3\le w\le\frac{1}{2}C_P\rho\eta Av_{rated}^3$, then

$$F_{P_t}(w)=1+\left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}-\left(1+\left(\frac{(2w)^{\frac{1}{3}}-\gamma(C_P\rho\eta A)^{\frac{1}{3}}}{\beta(C_P\rho\eta A)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k}.$$

If $\frac{1}{2}C_P\rho\eta Av_{rated}^3<w<+\infty$, then $F_{P_t}(w)=1$.

Proof: *For* $w<0$, it is obvious that $F_{P_t}(w)=P(W<w)=0$. It is evident that

$$P(W\le 0)\ =P(v\le v_{cut-in})+P(v>v_{cut-off})=F_V(v_{cut-in})+1-F_V(v_{cut-off})$$
$$=1-\left(1+\left(\frac{v_{cut-in}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}+\left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

Therefore $w\le 0$, we have that

$$F_{p_t}(w)=1-\left(1+\left(\frac{v_{cut-in}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}+\left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

When $\frac{1}{2}C_P\rho\eta Av_{cut-in}^3\le w\le\frac{1}{2}C_P\rho\eta Av_{rated}^3$, noting $w=\frac{1}{2}C_P\rho\eta Av^3$, we get that

$$v_{cut-in}\le v\le v_{rated}.$$

By Theorem 1, it is following that

$$P(W\le w)=P(W\le 0)+P\left(\frac{1}{2}C_P\rho\eta Av_{cut-in}^3\le\frac{1}{2}C_P\rho\eta Av^3\le w\right)$$

$$= P\left(W \le 0\right) + P\left(v_{cut-in} \le v \le \left(\frac{2w}{C_P \rho \eta A}\right)^{\frac{1}{3}}\right)$$

$$= 1 + \left(1 + \left(\frac{v_{cut-off} - \gamma}{\beta}\right)^{\alpha}\right)^{-k} - \left(1 + \left(\frac{(2w)^{\frac{1}{3}} - \gamma\left(C_P \rho \eta A\right)^{\frac{1}{3}}}{\beta\left(C_P \rho \eta A\right)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k}.$$

Therefore for $0.5 C_P \rho \eta A v_{cut-in}^3 \le w \le 0.5 C_P \rho \eta A v_{rated}^3$, we get that

$$F_{P_t}\left(w\right) = 1 + \left(1 + \left(\frac{v_{cut-off} - \gamma}{\beta}\right)^{\alpha}\right)^{-k} - \left(1 + \left(\frac{(2w)^{\frac{1}{3}} - \gamma\left(C_P \rho \eta A\right)^{\frac{1}{3}}}{\beta\left(C_P \rho \eta A\right)^{\frac{1}{3}}}\right)^{\alpha}\right)^{-k}.$$

When $0.5 C_P \rho \eta A v_{rated}^3 < w < +\infty$, it is obvious that $F_{P_t}\left(w\right) = 1$.

***Theorem* 10** Let the wind speed of wind turbine with power coefficient C_P and efficiency coefficient η produce power P_t, then the mean of P_t is given by

$$E\left(P_t\right) = \frac{1}{2} C_P \rho \eta A \int_{v_{cut-in}}^{v_{rated}} v^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1 + \left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv$$

$$+ \frac{1}{2} C_P \rho \eta A v_{v_{rated}}^3 \left(\left(1 + \left(\frac{v_{rated} - \gamma}{\beta}\right)^{\alpha}\right)^{-k} - \left(1 + \left(\frac{v_{cut-off} - \gamma}{\beta}\right)^{\alpha}\right)^{-k}\right).$$

Proof: By the definition of mathematical expectation, we get the following

$$E\left(P_t\right) = \int_{-\infty}^{+\infty} w\, d\left(F_{P_t}\left(w\right)\right)$$

$$= \int_{v_{cut-in}}^{v_{rated}} \frac{1}{2} C_P \rho \eta A v^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1 + \left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv$$

$$+ \int_{v_{rated}}^{v_{cut-off}} \frac{1}{2} C_P \rho \eta A v_{v_{rated}}^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1 + \left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv$$

$$= \frac{1}{2} C_P \rho \eta A \int_{v_{cut-in}}^{v_{rated}} v^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1 + \left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv$$

$$+ \frac{1}{2} C_P \rho \eta A v_{v_{rated}}^3 \left(\left(1 + \left(\frac{v_{rated} - \gamma}{\beta}\right)^{\alpha}\right)^{-k} - \left(1 + \left(\frac{v_{cut-off} - \gamma}{\beta}\right)^{\alpha}\right)^{-k}\right).$$

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4. Turbine site matching

In [15], the Capacity Factor is defined as

$$C_F = \frac{E(P_t)}{\frac{1}{2}C_P \rho \eta A v_{rated}^3}.$$

Suppose we have chosen a location where the wind speed V for a given month has a $Burr(\alpha, \beta, \gamma, k)$ distribution. Then C_F is given as

$$C_F = \frac{1}{v_{rated}^3}\int_{v_{cut-in}}^{v_{rated}} v^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv$$
$$+\left(1+\left(\frac{v_{rated}-\gamma}{\beta}\right)^{\alpha}\right)^{-k} - \left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

Furthermore, supposing a manufacturer can manufacture wind turbines with fixed Cut-in and Cut-off speeds represented by v_{cut-in} and $v_{cut-out}$ respectively, but can adjust the rated wind speed $v_{rated} \in [v_{cut-in}, v_{cut-out}]$. Suppose also that the power coefficient C_P, the efficiency coefficient η and the rotor area A are independent of v_{rated}. What should be the rated wind speed if we want the capacity factor C_F to be maximized? What should be the rated wind speed if we want to maximize the average power of the wind turbine $E(P_t)$? Are these two wind speeds equal? These questions can be answered by using the calculus [16]. Define a function

$$H(v) = \frac{1}{v^3}\int_{v_{cut-in}}^{v} v^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv$$

$$+\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k} - \left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}.$$

Therefore, we have that $H'(v) = \frac{-3}{v^4}\int_{v_{cut-in}}^{v} v^3 \frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1} dv.$

It is obvious that the last expression has a root precisely at $v = v_{cut-in}$. For $v > v_{cut-in}$, $H'(v) < 0$. We get that the capacity factor is maximized for $v_{rated} = v_{cut-in}$. In the following, we compute the value v_{rated} which maximizes $E(P_t)$. Define a function

$M(v):\left[v_{cut-in}, v_{cut-off}\right) \to$ ¡ .

$$M(v)=\int_{v_{cut-in}}^{v}\frac{1}{2}C_P\rho\eta Av^3\frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1}dv$$
$$+\frac{1}{2}C_P\rho\eta Av^3\left(\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k}-\left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}\right).$$

Hence, we get that

$$M'(v)==\frac{3}{2}C_P\rho\eta Av^2\left(\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k}-\left(1+\left(\frac{v_{cut-off}-\gamma}{\beta}\right)^{\alpha}\right)^{-k}\right).$$

For $v\in\left(v_{cut-in}, v_{cut-off}\right)$, it is obvious that $M'(v)>0$. We can conclude that

$$\max_{\left[v_{cut-in}, v_{cut-off}\right]}M(v)=\int_{v_{cut-in}}^{v_{cut-off}}\frac{1}{2}C_P\rho\eta Av^3\frac{\alpha k}{\beta}\left(\frac{v-\gamma}{\beta}\right)^{\alpha-1}\left(1+\left(\frac{v-\gamma}{\beta}\right)^{\alpha}\right)^{-k-1}dv.$$

We get that the average power of the wind turbine is maximized for $v_{rated}=v_{cut-off}$.

These results indicate that by having lowly rated wind turbines, we can maximize the capacity factor at the cost of minimizing expected power output. Conversely, we can increase expected power output at the cost of decreasing the capacity factor.

5. Conclusion

In this study, some mathematical properties of four parameters Burr distribution are addressed, and the features of wind power density are obtained by analyzing dependency between wind power density and wind speed density with four parameters Burr distribution. According to the wind power density, we can use the concept of mean to obtain mathematical property which is an important index that is associated with the estimation of potential wind energy. This study presents mathematical proofs of such estimates which is required for accurate wind power production simulation with a specific cut-in, cut-off and rated speeds, and which can also compute the capacity factor.

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