IOP Conf. Series: Materials Science and Engineering **317** (2018) 012029 doi:10.1088/1757-899X/317/1/012029

Dynamics of Longitudinal Impact in the Variable Cross-Section Rods

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Abstract. Dynamics of longitudinal impact in rods of variable cross-section is considered. Rods of various configurations are used as elements of power pulse systems. There is no single method to the construction of a mathematical model of longitudinal impact on rods. The creation of a general method for constructing a mathematical model of longitudinal impact for rods of variable cross-section is the goal of the article. An elastic rod is considered with a cross-sectional area varying in powers of law from the longitudinal coordinate. The solution of the wave equation is obtained using the Fourier method. Special functions are introduced on the basis of recurrence relations for Bessel functions for solving boundary value problems. The expression for the square of the norm is obtained taking into account the orthogonality property of the eigen functions with weight. For example, the impact of an inelastic mass along the wide end of a conical rod is considered. The expressions for the displacements, forces and stresses of the rod sections are obtained for the cases of sudden velocity communication and the application of force. The proposed mathematical model makes it possible to carry out investigations of the stress-strain state in rods of variable and constant cross-section for various conditions of dynamic effects.

1. Introduction

The research of dynamic processes is carried out using the elastic rod model for a wide range of objects, such as: elements of drilling equipment [1-3], buildings and structures [4], power impulse systems [5, 6], etc. Especially urgent is the problem of longitudinal impact for power pulse systems, which are used in machines that perform various technological processes: punching, forging, destruction of rocks, concrete coatings, piling, etc. The problem of increasing the productivity of impact machines includes not only increasing the power, but also increasing the efficiency of energy transfer to the process area. The latter is achieved, among other things, on the basis of studying the process of formation of deformation waves by strikers of various geometries and searching for strings structures that create deformation waves with rational parameters [5].

At considering the various models of longitudinal collision of bodies [5, 7], the wave model of Saint-Venant's shock is taken as the basis, since it most fully reflects the real dynamic processes in colliding bodies, and for its practical implementation the d'Alembert method is used. The exact solution of the wave model is given by the Fourier method [8].

At solving the equation of longitudinal oscillations of variable cross-section rods, in the case of impact, arise certain mathematical difficulties, for example, the given equation is an equation with variable coefficients, the orthogonality of eigen functions, etc. Therefore, for solving the such problems are using various simplifying hypotheses. Thus, in [9] systems with distributed parameters are replaced by single-mass systems (Cox's theory) or various refinements [10], or are using an approximation of the dynamic deformations forms by static [11, 12], and in paper [13] is used the method of averaging variable coefficients. In [7, 14] are considered the models of longitudinal impact of various shapes rods with the using of their surface approximation by successively conjugate cylindrical sections. Using the operational calculus in work [3], is carried out an analysis of the using effectiveness of application of elements of drilling equipment (picks, drill rods) with different configurations, and for each of the schemes under consideration, a custom mathematical model is being constructed. The longitudinal impact of rods with conical and hyperbolic forms is considered in [15, 16], solutions of the wave equation for which are obtained in elementary functions, and in [17] dynamical processes of longitudinal impact in hollow rods of conical shape are studied using Bessel functions.

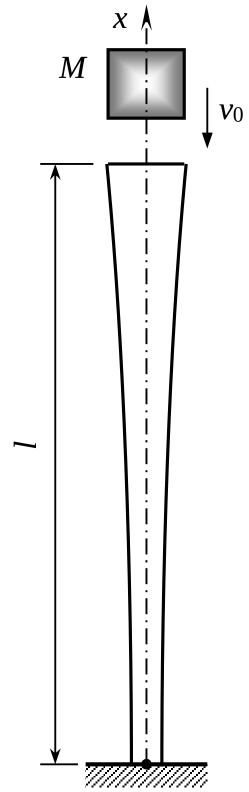

Figure 1. Scheme of longitudinal impact of a complex configuration rod.

Thus, based on the fact that in the theory of calculations, at the present time, there is no single approach to constructing a mathematical model of the longitudinal impact of complex configuration rods, the search and development of new solutions to longitudinal impact problems is an actual scientific and practical problem.

2. Development of a mathematical model

As a mathematical model of the object under consideration, we take an elastic rod of length l, the distributed mass m and the cross-sectional area F, which varies according to the exponent law from the longitudinal coordinate x:

$$m = \gamma F \;;\; F = F_2 z^{\mu};\; z = (1-k)\frac{x}{l} + k\,,\;\; k = \frac{h_1}{h_2}\;\; (0 < k < 1) \tag{1}$$

when γ – material density, F_2 – the cross-sectional area of the larger base of the rod, h_1 и h_2 – the parameters of the cross dimensions of the upper and lower sections are determined by the geometry of the rod cross-section of the rod (the radius of the cross-sections for conical structures), the exponent μ depends on the rod configuration, for example, for conical tubes $\mu = 1$, and for a conic rod of solid cross section $\mu = 2$ (figure 1).

Equation of longitudinal displacements $u(x,t)$, taking into account the accepted notation, will have the view [17].

$$z^{\mu}\frac{\partial^2 u}{\partial z^2} + \mu z^{\mu-1}\frac{\partial u}{\partial z} = \frac{\gamma l^2}{E(1-k)^2} z^{\mu}\frac{\partial^2 u}{\partial t^2} + \frac{l^2}{EF_2(1-k)^2}p(z,t), \tag{2}$$

when E – elasticity modulus, $p(x,t)$ – external load.

Let us find the general solution of equation (2) at $p(z,t) \equiv 0$ (the case of free oscillations).

Separating the variables, obtain an equation for the eigen functions whose solution has the view [18].

$$Z_n(z) = z^{\nu}\left(C_1 J_{\nu}(\lambda_n z) + C_2 Y_{\nu}(\lambda_n z)\right) \tag{3}$$

when

$$\nu = \frac{1-\mu}{2}\;;\; \lambda_n = \frac{\omega_n l}{(1-k)}\sqrt{\frac{\gamma}{E}}\,,$$

ω_n – oscillation eigen frequencies, $J_{\nu}(z)$ и $Y_{\nu}(z)$ – Bessel functions.

For the convenience of solving problems with different boundary conditions, we introduce the following notation for functions:

$$A_n(z)=\frac{\pi}{2}\lambda_n k\left(J_\nu(\lambda_n z)Y_{\nu-1}(\lambda_n k)-Y_\nu(\lambda_n z)J_{\nu-1}(\lambda_n k)\right);$$

$$B_n(z)=\frac{\pi}{2}\lambda_n k\left(Y_\nu(\lambda_n z)J_\nu(\lambda_n k)-J_\nu(\lambda_n z)Y_\nu(\lambda_n k)\right);$$

$$C_n(z)=\frac{\pi}{2}\lambda_n k\left(J_{\nu-1}(\lambda_n z)Y_{\nu-1}(\lambda_n k)-Y_{\nu-1}(\lambda_n z)J_{\nu-1}(\lambda_n k)\right);$$

$$D_n(z)=\frac{\pi}{2}\lambda_n k\left(Y_{\nu-1}(\lambda_n z)J_\nu(\lambda_n k)-J_{\nu-1}(\lambda_n z)Y_\nu(\lambda_n k)\right).$$

In point $z=k$ the presented functions take the values:

$$A_n(k)=D_n(k)=1;\; B_n(k)=C_n(k)=0.$$

Using the adopted dependencies, arbitrary constants in (3) can be expressed in terms of displacement u_0 and force N_0 in the zero section, then the expressions for displacements and longitudinal forces for an arbitrary waveform can be represented as:

$$u_n(z)=Z_n=z^\nu\left(u_0 k^{-\nu}A_n(z)+\frac{N_0 l}{EF_2(1-k)}\frac{k^{\nu-1}}{\lambda_n}B_n(z)\right); \tag{4}$$

$$N_n(z)=\frac{EF_2(1-k)}{l}z^{1-2\nu}Z_n'=z^{1-\nu}\left(u_0\frac{EF_2(1-k)}{l}k^{-\nu}\lambda_n C_n(z)+N_0 k^{\nu-1}D_n(z)\right). \tag{5}$$

On the basis of the reciprocity theorem [19], at the absence of concentrated masses, the eigen functions will be orthogonal with weight $\rho(z)=z^{1-2\nu}$. To find the square of the eigen functions norm, we first proceed in the same way as in [17] for eigen functions with different indices.

$$\left(\lambda_n^2-\lambda_m^2\right)\int_k^1 z^{1-2\nu}Z_m Z_n dz=Z_m\left(z^{1-2\nu}Z_n'\right)-Z_n\left(z^{1-2\nu}Z_m'\right)\Big|_k^1.$$

Passing to the limit at $m\to n$, are getting

$$2\lambda_n\int_k^1 z^{1-2\nu}Z_n^2(z)dz=\frac{\partial Z_n}{\partial\lambda_n}\left(z^{1-2\nu}Z_n'\right)-Z_n\frac{\partial\left(z^{1-2\nu}Z_n'\right)}{\partial\lambda_n}\Bigg|_k^1.$$

Having determined the partial derivatives by λ_n, a square of the eigen functions norm find in the form

$$\Delta_n^2=\frac{1}{2}\left(z^{2-2\nu}Z_n^2+\frac{z^{2\nu}}{\lambda_n^2}\left(z^{1-2\nu}Z_n'\right)^2-\frac{2\nu-1}{\lambda_n^2}\left(z^{1-2\nu}Z_n'\right)Z_n\right)\Bigg|_k^1. \tag{6}$$

Now consider the case of forced oscillations, this requires finding a solution $u_2(z,t)$ of non-linear equation

$$\left(z^{1-2\nu}u'\right)'-\beta z^{1-2\nu}\ddot{u}=\frac{\beta}{\gamma F_2}p(z,t),\quad \beta=\frac{\gamma l^2}{E(1-k)^2} \tag{7}$$

The equation solution (7) can be represented as a row by eigen functions

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$$u_2(z,t)=\sum_{n=1}^{\infty} w_n(t) Z_n(z).$$

Substituting this row into (7) and taking into account relation $\lambda_n^2=\beta\omega_n^2$, are getting

$$\sum_{n=1}^{\infty}\left(\ddot{w}_n+\omega_n^2 w_n\right) z^{1-2\nu} Z_n=-\frac{p(z,t)}{\gamma F_2} \tag{8}$$

Applying the Fourier method to equation (8), taking into account the orthogonality property of the eigen functions with weight, obtain the equation with respect to the coefficients w_n

$$\ddot{w}_n+\omega_n^2 w_n=-\frac{1-k}{\gamma F_2 l \Delta_n^2} \int_k^1 p(z,t) Z_n(z) dz \tag{9}$$

As an example, consider the scheme of mass impact M on the upper end of the rod, while the mass is held for some time by the rod, and its lower end is rigidly limited (figure 1), such a circuit can simulate a stress-strain state in waveguides of power pulse systems or in piles in the process of their driving. For the adopted scheme, the movement of the bottom end will be $u_0=0$, the boundary condition on the second end of the rod

$$N(l,t)=-M\ddot{u}(l,t) \tag{10}$$

initial conditions

$$u(x,0)=0\,;\ \dot{u}(x,0)=-v_0 e(x-l) \tag{11}$$

when *e(x)* – a unit function.

The equation for displacements and longitudinal forces for an arbitrary shape is obtained in the form

$$u_n(z)=\frac{N_0 l}{EF_2(1-k)} Z_n=\frac{N_0 l}{EF_2(1-k)} \frac{k^{\nu-1}}{\lambda_n} z^{\nu} B_n(z);$$
$$N_n(z)=N_0 z^{1-2\nu} Z_n'=N_0 z^{1-\nu} k^{\nu-1} D_n(z).$$

From the boundary condition (10) obtain the equation for finding the eigen values

$$D_n(1)-\zeta \frac{1-k^{2-2\nu}}{2(1-\nu)} \lambda_n B_n(1)=0 \tag{12}$$

here $\zeta=M/M_0$, M_0 – the road mass.

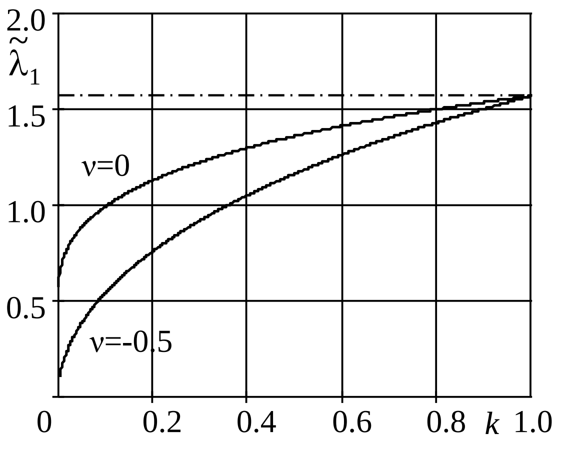

Figure 2. Depending $\tilde{\lambda}_1$ from the value of the relative cross dimension parameter *k*.

Taking into account the fact, that in some approximate methods of dynamic calculation [9, 10] is using the value of the first eigen frequency, we estimate the effect of the parameter on the values of the first eigen value of (12)

$$\omega_1=\frac{\tilde{\lambda}_1}{l}\sqrt{\frac{E}{\gamma}},\ \tilde{\lambda}_n=\lambda_n(1-k),$$

the frequencies of the rods of constant cross section are determined in the same way [8]. Numerical studies of dependence $\tilde{\lambda}_1$ on the parameter value k for conic hollow roads (ν=0) and solid (ν=-0,5) sections, without taking into account the concentrated mass ($\zeta=0$), are presented at figure

2. It can be seen from the graph that for the case $k \approx 1$ the value $\tilde{\lambda}_1$ coincides with the known value $\tilde{\lambda}_1 = \pi/2 \approx 1{,}571$ [8] (is indicated by the dot-dash line on the graph figure 2).

Given the presence of the concentrated mass on the upper end of the rod (figure 1), the eigen functions will be orthogonal with the weight

$\rho(z) = z^{1-2\nu} + \zeta \dfrac{1-k^{2-2\nu}}{2(1-\nu)}\delta(z-1)$, where $\delta(z)$ – Dirac’s delta function. Then from the relation (6) the square of the eigen functions norm will be determined by the formula

$$\Delta_n^2 = \frac{1}{2\lambda_n^2}\left(\left(1+\zeta\frac{1-k^{2-2\nu}}{2(1-\nu)}\left(\zeta\frac{1-k^{2-2\nu}}{2(1-\nu)}\lambda_n^2 - 2\nu + 2\right)\right)k^{2-2\nu}B_n^2(1) - k^{2\nu}\right).$$

First, we find the movements and forces from the the speed reporting. From the first initial condition (11) it follows that

$$u_v(z,t) = \sum_{n=1}^{\infty}\Phi_n Z_n(z)\sin\omega_n t \tag{13}$$

If the second initial condition (11) is satisfied, then in expression (13) the coefficients Φ_n will take the view

$$\Phi_n = -\frac{v_0 l}{c}\zeta\frac{\left(1-k^{2-2\nu}\right)k^{\nu-1}B_n(1)}{4(1-\nu)(1-k)\lambda_n^2\Delta_n^2},$$

where $c = \sqrt{E/\gamma}$ - longitudinal wave velocity of the rod [8].

Dependences for displacements, forces and stresses are obtained in the view

$$u_v(z,t) = -\frac{v_0 l}{c}\zeta\frac{\left(1-k^{2-2\nu}\right)k^{2\nu-2}}{4(1-\nu)(1-k)}\sum_{n=1}^{\infty}\frac{B_n(1)}{\lambda_n^3\Delta_n^2}z^{\nu}B_n(z)\sin\omega_n t \tag{14}$$

$$N_v(z,t) = -\frac{v_0}{c}EF_2\zeta\frac{\left(1-k^{2-2\nu}\right)k^{2\nu-2}}{4(1-\nu)}\sum_{n=1}^{\infty}\frac{B_n(1)}{\lambda_n^2\Delta_n^2}z^{1-\nu}D_n(z)\sin\omega_n t \tag{15}$$

$$\sigma_v(z,t) = -\frac{v_0}{c}E\zeta\frac{\left(1-k^{2-2\nu}\right)k^{2\nu-2}}{4(1-\nu)}\sum_{n=1}^{\infty}\frac{B_n(1)}{\lambda_n^2\Delta_n^2}z^{\nu}D_n(z)\sin\omega_n t \tag{16}$$

Taking into account to the sudden weight application Mg (9) will have the view

$$\ddot{w}_n + \omega_n^2 w_n = -\frac{Mg(1-k)}{\gamma F_2 l\Delta_n^2}Z_n(1),$$

and their solution

$$w_n(t) = -u_2\frac{2\nu k^{\nu-1}B_n(1)}{\Delta_n^2\lambda_n^3\left(1-k^{2\nu}\right)}(1-\cos\omega_n t);\; u_2 = \frac{Mgl\left(1-k^{2\nu}\right)}{2\nu(1-k)EF_2}.$$

Thus, in the final form obtain the equations of displacements, forces and stresses from the application of the load

$$u_p(z,t) = -u_2\frac{2\nu k^{2\nu-2}}{\left(1-k^{2\nu}\right)}\sum_{n=1}^{\infty}\frac{B_n(1)}{\Delta_n^2\lambda_n^4}z^{\nu}B_n(z)(1-\cos\omega_n t) \tag{17}$$

$$N_p(z,t) = -Mgk^{2\nu-2}\sum_{n=1}^{\infty}\frac{B_n(1)}{\Delta_n^2\lambda_n^3}z^{1-\nu}D_n(z)(1-\cos\omega_n t) \tag{18}$$

$$\sigma_p(z,t) = -\frac{Mg}{F_2} k^{2\nu-2} \sum_{n=1}^{\infty} \frac{B_n(1)}{\Delta_n^2 \lambda_n^3} z^{\nu} D_n(z)(1-\cos\omega_n t) \tag{19}$$

Introduce the notation for dimensionless quantities:

$\tau = \frac{t}{l}\sqrt{\frac{E}{\gamma}}$ – dimensionless time, $\tilde{\sigma}_v(z,t) = \frac{\sigma_v(z,t)c}{v_0 E}$; $\tilde{\sigma}_p(z,t) = \frac{\sigma_p(z,t)F_2}{Mg}$ – relative stresses, $\tilde{u}_p(z,t) = \frac{u_p(z,t)}{u_2}$ – relative displacement.

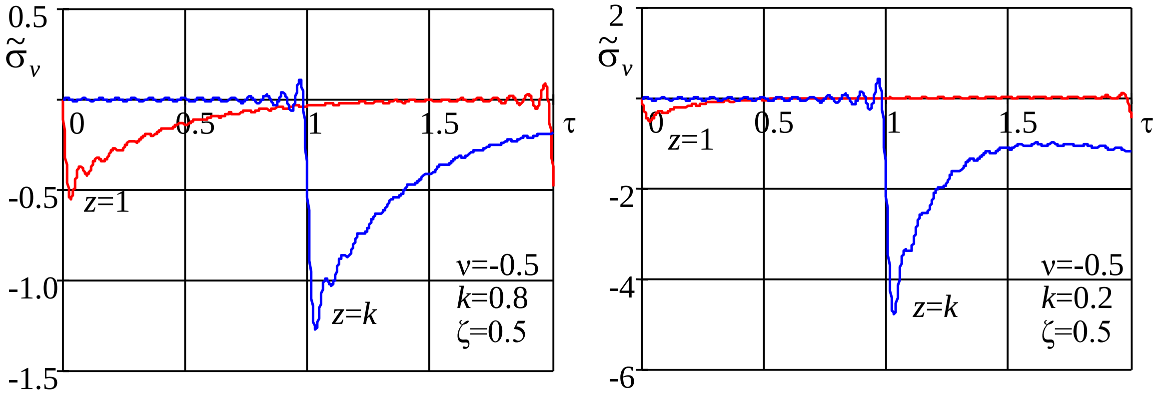

Figure 3. The relative magnitude of the stresses in the end sections of the rods of different configurations from the velocity reporting.

At figure 3 are shown the dependencies of stresses $\tilde{\sigma}_v$ from time τ, caused by communication to the upper end of the speed rod from the falling load ($\zeta = 0{,}5$) in the shock end ($z = 1$) and at the lower end ($z = k$), for conical shaped rods of solid section ($\nu = -0{,}5$),with the ratio of the radiuses of the upper and lower sections, respectively: $k = 0{,}8$ и $k = 0{,}2$. Analysis of stresses and stresses for various parameters of the scheme showed that for rods with a decrease in the value of the relative cross dimension *K*, the stresses at the lower end increase, but not in proportion to the decrease in the cross-sectional area, but to a much lesser extent, i.e. the force decreases. Thus, for waveguides of power pulse systems, it is expedient to use rods of constant cross section with sharpening only in the area of the working part, using of rods with a decreasing cross section will lead to a decrease in the efficiency of transferring the force pulse to the working zone.

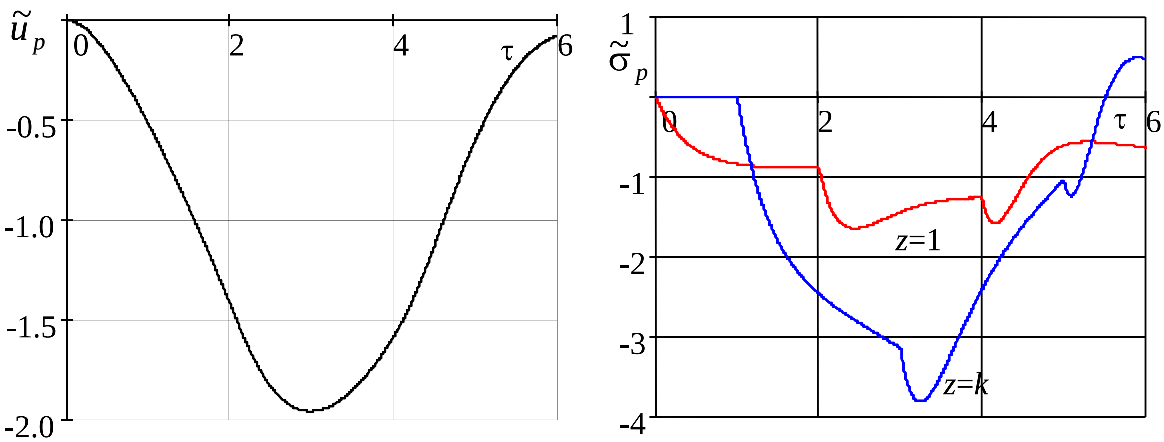

Figure 4. Relative values of displacements and stresses in the end sections of the rod with parameters ν=-0.5; *k*=0.8; ζ=0.5, from the load application.

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At figure 4. are presented the relative displacements $\tilde{u}_p$ and stresses $\tilde{\sigma}_p$ depending on time τ, caused by a sudden application to the upper end of the power shaft, for a conical shaped rod of solid section ($\nu = -0{,}5$), with the ratio of the radiuses of the upper and lower sections $k = 0{,}8$. The study showed that different parameters of the scheme do not significantly affect the values of the maximum relative displacements and forces, the stresses in the lower section increase in proportion to the reduction in the cross-sectional area. Those, in systems where the transfer of the pulse is mainly due to the application of the load, and not the speed, in this case the configuration of the rod will not affect the transmission efficiency of the power pulse.

3. Conclusions

The proposed mathematical model of the rod longitudinal impact with complex configuration allows both direct investigation of the stress-strain state and, in contrast to numerical methods and means of object modeling, the analytical solution makes it possible to estimate the degree of influence of various model parameters on the unknown quantities. The presented algorithm of calculation can be put in the basis of CAD for imitating modeling of complex mechanical systems. It should also be noted that all the parameters of the model are determined and at the value $\nu = 0{,}5$ that corresponds to the rods of the constant section, so can use the unified approach to the construction of rod models with various configurations

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