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Isomorphic observers of the linear systems state

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Abstract. The paper suggests isomorphic observers as a new type of observers. It is shown that isomorphic observers provide the maximum achievable quality of monitoring and filtering for linear systems. The techniques of synthesis of such observers are discussed. Isomorphic observers are multiplicative in form. This permits us to subdivide the general problem into separate tasks of the observer and filter synthesis. The means to achieve robustness and accuracy for isomorphic observers at high levels noises and parametric perturbations are also discussed. The example of the synthesis is presented.

1. Introduction

The most famous are two types of observers: the ones based on Kalman filter [1–4] and Luenberger observers [2, 3, 5]. They are applied mainly to linear systems and are asymptotic [1–5]. At the same time these observers carry out the function of signal filtering. However, the heuristic compromise has to be established between the qualities of observation and filtering. The lack of robustness of these observers is also eliminated heuristically [2, 3, 6].

Modern methods of observer synthesis are complex and poorly formalized. This applies to linear systems, which are often used in practice. Synthesis of observers for nonlinear systems is still more sophisticated [2, 3, 7–11].

The work [12] formally defines the structure of the new type of observers for general-form linear systems. These observers are called isomorphic [12, 13]. Based on the results of the works [12, 13], the techniques of synthesis of such observers are discussed. Isomorphic observers are multiplicative in form. This permits us to subdivide the general task into separate tasks of the observer and filter synthesis. It is demonstrated that isomorphic observers provide the linear systems with the maximum achievable quality of monitoring and filtering.

2. Linear isomorphic observers

The general-form linear system is considered

$$\begin{aligned} &\dot{X} = AX, X(0) = X_0, \\ &Z = HX, X \subset R^n, Z \subset R^m. \end{aligned} \tag{1}$$

In the Laplacian form (1) it is presented as follows

$$\begin{cases} X(p) = \Phi X(p), \\ Z(p) = HX(p). \end{cases} \tag{2}$$

Here $\Phi(p)=(pI–A)^{-1}$ is the fundamental matrix, I – the identity matrix, $Z(p)$ – the vector of direct measurements. It is required [1, 2] to obtain the estimate of the vector $X(0)=X_0$. The matrix $\Phi(p)$ is always inverse, i.e. is an isomorphism. If (2) is observable, then there is a commutative diagram [12, 13], shown in Fig. 1. If $m \geq n$, the observer S_N (2) is determined by the formula [12, 13]

$$S_N = \Phi^{-1}(p)H^{-1} = (pI–A)H^{-1} \tag{3}$$

$$\begin{tikzcd} \boldsymbol{X_0} \arrow[r, "\boldsymbol{\Phi}"] & \boldsymbol{X} \arrow[r, "\boldsymbol{H}"] & \boldsymbol{Z} \arrow[ll, bend left, "\boldsymbol{S_N}"] \end{tikzcd}$$

Figure 1. The projections composition chart for the observed linear system

Observers (3) are isomorphic [12, 13]. In the commutative diagram they establish a one-to-one correspondence between Z and X_0 with accuracy up to isomorphism Φ. It is shown [12, 13] that, for observed object, if $m \geq n$ there are always inverse $(S_N)^{-1}$ and H^{-1}. At the same time the matrices S_N and H can be rectangular and are non-invertible outside the chart. If the object is observable, and $m < n$, it is necessary to find another isomorphism, with the precision to which the observer can be calculated. In this case, it is always possible to transform (1) into the form with the invertible (isomorphic) matrix H^*. It connects the same outputs Z of the sensors with the vector of measured coordinates $X^* \subset X$, as in (1). The conversion is performed by expressing the unobserved coordinates through the measured. The possibility of such a transformation may serve as a criterion of observability.

Let the system be observable and $m \geq n$. We require that the value $\tilde{X}_0$ generated at the output of the observer satisfies the condition $\tilde{X}_0 \equiv X_0$. There are possible noises in the input signal X_0 and in the signal $\tilde{X}_0$. For their filtering we use the filter $W_\Phi =(pI–C)^{-1}D$, where C and D are matrices. The filter is selected by engineering quality criteria [14]. The equations (1) with the filter will take the form of

$$\begin{aligned} &X(p) = \Phi X_0, \\ &Z(p) = HX(p), \\ &\tilde{X}_0(p) = S_N Z(p), \\ &\hat{X}_0(p) = W_\Phi \tilde{X}_0(p). \end{aligned} \tag{4}$$

In the simplest case, if $\hat{X}_0 \equiv \tilde{X}_0$, $W_\Phi=I$, where I is the identity matrix. Fig. 2 shows the diagram for (4), which allows to transform (3) into the form

$$S_N=I\Phi^{-1}(p)H^{-1}=(pI–A)H^{-1},\ (S_N)^{-1}=H\Phi(p)I=H\Phi(p)=H(pI–A)^{-1}, \tag{5}$$

where $(S_N)^{-1}$ and H^{-1} satisfy the conditions of [12, 13]:

$$\boldsymbol{I}_{X_0}^{left} = S_N (S_N)^{-1},\ \boldsymbol{I}_{X_0}^{right} = \Phi^{-1}\Phi,\ \boldsymbol{I}_{X_0}^{left} = \boldsymbol{I}_{X_0}^{right} = \boldsymbol{I}\ ,$$

$$\boldsymbol{I}_Z^{left} = HH^{-1}\ ,\ \boldsymbol{I}_Z^{right} =(S_N)^{-1}S_N,\ \boldsymbol{I}_Z^{left} = \boldsymbol{I}_Z^{right} = I_Z\ , I_Z I_Z= I_Z\ , I_Z = (I_Z)^{-1}. \tag{6}$$

$$\begin{tikzcd} X_0 \arrow[r, "\boldsymbol{\Phi}"] \arrow[rrr, bend right, "\boldsymbol{I}"'] & X \arrow[r, "\boldsymbol{H}"] & Z \arrow[r, "\boldsymbol{S_N}"] & \tilde{X}_0 \arrow[r, "\boldsymbol{W_\Phi}"] & \hat{X}_0 \end{tikzcd}$$

Figure 2. The projections composition chart for the system (4)

Here $I_{X_0}^{left}$, $I_{X_0}^{right}$ are the left and the right units on $\tilde{X}_0 \equiv X_0$, I_Z^{left} and I_Z^{right} – the left and the right units on *Z*. Unit on $\tilde{X}_0 \equiv X_0$ is equa lto the identity matrix *I*. In general [12, 13] I_Z^{left} and I_Z^{right} can be non-trivial matrices in the ordinary sense, but such that (6) holds true. The equations (5) and (6) allow us to determine the isomorphic observer S_N with the accuracy up to isomorphism Φ. If the model (2) corresponds exactly to the object (under the object we understand the object of observation, together with the sensors), the monitoring circuit with feedback (FB) S_N (see Fig. 1, 2) acquires a “linear” shape (see Fig.3). Taking (5) into account, it takes the form shown in Fig. 4.

$$X_0 \xrightarrow{\Phi} X \xrightarrow{H} Z \xrightarrow{S_N} \tilde{X}_0 \xrightarrow{W_\Phi} \hat{X}_0$$

Fig. 3. "Linear" circuit with isomorphic observer

$$X_0 \xrightarrow{\textbf{Object}} Z \xrightarrow{S_N} \tilde{X}_0 \xrightarrow{W_\Phi} \hat{X}_0$$

Fig. 4. "Linear" circuit of real object monitoring

Such a "linear" circuit meets the condition $S_N(S_N)^{-1}=I$ of precise tracking $\tilde{X}_0 \equiv X_0$ at any point in time. S_N is the perfect observer: based on *Z* data it allows to observe the state X_0 instantly, fully, and accurately and hence also the state *X* at the time t under certain conditions $t_0 \to t$. The observer S_N is multiplicative, as it requires just matrix multiplication. As it can be seen from (5), Fig. 3 and Fig. 4, the result of the observation X_0 is completely determined by the choice of filter. The observer and the filter are synthesized separately. For the ideal observer the filter is formed under the given requirements to filtration quality. In this sense, the isomorphic observer is the ultimate in quality of signal X_0 monitoring and filtering. One should consider $W_\Phi=(pI-C)^{-1}D$ [13] as a type of filter, which, together with S_N forms physically realizable structures of monitoring and filtering.

In case of differences between the system and the model (2), (4), the result of the monitoring and filtering may be unsatisfactory. Then the "linear" circuit will need to be covered by additional feedback, as shown in Fig. 5.

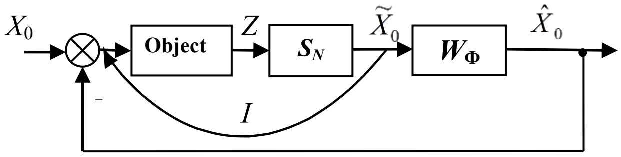

Figure 5. Robust monitoring circuit

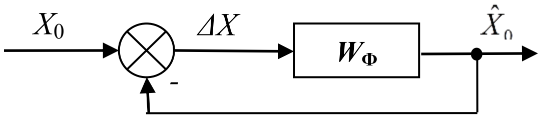

Figure 6. Robust circuit for the model, adequate to the object

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For the model (4), adequate to the object, the transfer function "object-observer" from X_0 to $\tilde{X}_0$ is close to $W_{OH} \approx I$. A closed circuit from X_0 to $\hat{X}_0$ (see Fig. 5) takes the form shown in Fig. 6. The transfer function of the closed circuit is $W = W_\Phi (I + W_\Phi)^{-1}$. That is, the characteristics of the robust circuit, as well as the "linear" one, are defined only by the structure of the filter. The astatism of the circuit is ensured, like in [15, 16], by the choice of W_Φ with high gain coefficient.

3. Example

Let $m<n$ and just one output be available for measurement in (2)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -4 & -1{,}14 \\ 1 & -0{,}62 \end{bmatrix}, \mathrm{H} = \begin{bmatrix} h_{11} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{7}$$

Write (2) with regard to (7) in scalar form

$$\begin{cases} x_1 = (p - a_{11})^{-1} a_{12} x_2 + (p - a_{11})^{-1} x_{10}, \\ x_2 = (p - a_{22})^{-1} a_{21} x_1 + (p - a_{22})^{-1} x_{20}, \\ z = h_{11} x_1. \end{cases} \tag{8}$$

By substitution of the second equation to the first one and some other evident substitutions, we will get the estimation $\tilde{x}_{20}$:

$$\tilde{x}_{20} = \left[(p - a_{11})\ a_{12}^{-1}(p - a_{22}) - a_{21} \right] h_{11}^{-1} z - a_{12}^{-1}(p - a_{22}) x_{10} = a_{20}^{-1} h_{11}^{-1} z + a_{10} x_{10}, \tag{9}$$

where $a_{20}^{-1} = [(p - a_{11})\ a_{12}^{-1}(p - a_{22}) - a_{21}], a_{10} = -a_{12}^{-1}(p - a_{22})$. The observer in (9) is calculated with accuracy up to isomorphism h_{11} using the formula

$$S_N = a_{20}^{-1} h_{11}^{-1}. \tag{10}$$

Thus,

$$\tilde{x}_{20} = a_{10} x_{10} + S_N z \tag{11}$$

In the observation equation (11), as well as in the Kalman filter, the updating sequence characterized by the second addend $S_N z$, is formed by mesurements. Modeling of the "linear circuit" (see Fig. 4) with the observer (10), (11) and a filter $W_\Phi = k_\Phi (T_\Phi p + 1)^{-1}$ with $k_\Phi = 1$ and $T_\Phi = 0{,}4$ has shown that the estimation $\hat{x}_2$ tracks the change x_2 accurately in accordance with the dynamics of the filter. With inaccurate knowledge of the elements a_{ij} in (7), (8) an error appears in the "linear" circuit in observation x_2. Closing the circuit with an additional feedback in accordance with Fig. 5–6 provides the robustness. Linear and robust circuits were also simulated with filters [13]

$$W_\Phi = \frac{1}{0.4p + 1} \cdot \frac{1}{1.5p + 1}$$ – for "linear" circuit, and

$$W_\Phi = \frac{1}{0.4p + 1} \cdot \frac{99}{1.5 * 100 p + 1}$$ – for a robust closed circuit.

Fig. 7 illustrates graphs of estimation and filtering of noisy input x_2 (figure 1), with the presence of normal white noise. The amplitude of the noise was 10% of the amplitude of the useful signal.

Graph 2 corresponds with the estimation $\hat{x}_2$ at the output of the "linear" circuit at the level of distortion of 40% of the parameter a_{11} in the object (7), (8). Graph 3 illustrates the estimation $\hat{x}_2$ at the output of robust circuit when the distortion of the parameter a_{11} is equal to 40%.

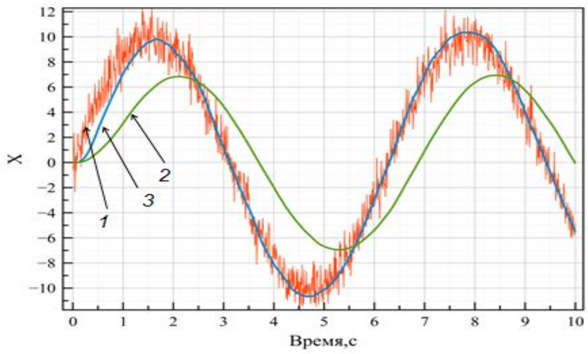

Figure 7. Estimation and filtering of noisy signal

It is seen that the "linear" circuit is subject to the influence of parametric perturbations, however it possesses a high-quality signal filtering. The circuit with an additional feedback possesses the property of robustness to significant variations (not known in advance) of the object parameters and a high-quality filtering of noises present in the input signal. If the deviations from the nominal values are known in advance (like systematic errors of the sensors), the "linear" circuit can be calibrated by selection of the filter parameters and elimination of their influence. The examples of estimating systematic errors in case of an aircraft are given in [17,18]. Additional feedback is needed in the case of significant deviations of the object parameters from those adopted in the model. The astaticism of the circuit can be adjusted by changing the gain coefficient of the filter.

So we conclude that in this paper the synthesis of isomorphic regulators is formalized to the maximum. High quality monitoring and filtering is ensured by filter selection taking into account the physical realizability of the system "observer-filter" and the desired properties of signal filtering. In that sense, isomorphic observers are "marginal" in terms of the achievable quality of estimation and filtering in comparison with well-known observers.

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