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Ideal flow theory for the double – shearing model as a basis for metal forming design

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Abstract. In the case of Tresca' solids (i.e. solids obeying the Tresca yield criterion and its associated flow rule) ideal flows have been defined elsewhere as solenoidal smooth deformations in which an eigenvector field associated everywhere with the greatest principal stress (and strain rate) is fixed in the material. Under such conditions all material elements undergo paths of minimum plastic work, a condition which is often advantageous for metal forming processes. Therefore, the ideal flow theory is used as the basis of a procedure for the preliminary design of such processes. The present paper extends the theory of stationary planar ideal flow to pressure dependent materials obeying the double shearing model and the double slip and rotation model. It is shown that the original problem of plasticity reduces to a purely geometric problem. The corresponding system of equations is hyperbolic. The characteristic relations are integrated in elementary functions. In regions where one family of characteristics is straight, mapping between the principal lines and Cartesian coordinates is determined by linear ordinary differential equations. An illustrative example is provided.

1. Introduction

Ideal plastic flows are those for which all material elements follow minimum work paths [1]. This theory has been fully developed for the constitutive equations comprising the Tresca yield criterion and its associated flow rule [2]. A comprehensive overview of the ideal flow theory has been provided in [3]. Recently, the existence of stationary planar ideal flow solutions has been proven for models of anisotropic and pressure – dependent plasticity [4, 5]. In particular, the model of anisotropic plasticity proposed in [6] has been adopted in [4] and the double slip and rotation model of pressure – dependent plasticity proposed in [7] has been adopted in [5]. The ideal flow theory is used as the basis of procedures for the direct preliminary design of forming processes. A number of such design solutions have been developed in [8 - 13]. In the case of stationary ideal flows, the distribution of plastic strain in the final product is uniform. Moreover, the distribution of the damage parameter is uniform for certain damage evolution laws [14, 15]. The present paper develops a method for constructing a class of stationary planar ideal solutions for the double – shearing model of pressure – dependent plasticity proposed in [16]. This model has been originally proposed for soils. However, it has been shown (see, for example, [17]) that it is adequate for some metallic materials as well.

2. Ideal flow conditions

In the case of the constitutive equations comprising the Tresca yield criterion and its associated flow rule, the principal stress and strain rate directions coincide. The ideal flow condition demands that the trajectories of the greatest (algebraically) principal stress (and strain rate) are fixed in the material [2]. In the case of plane strain deformation, the trajectories of both principal stresses (and strain rates) in planes of flow are fixed in the material. Therefore, the principal lines coordinate system is Lagrangian and solving the boundary value problem of plasticity reduces to a purely geometric problem [18]. In the case of stationary flows, the aforementioned general ideal flow condition requires that the trajectories of the greatest principal stress coincide with streamlines. Solving plane strain problems also reduces to a purely geometric problem [19]. A proof of the existence of stationary bulk planar ideal flows for the double slip and rotation model proposed in [7] has been given in [5]. However, no method for finding solutions has been developed. The model proposed in [7] is based on the Mohr - Coulomb yield criterion and a non-associated flow rule. The principal stress and strain rate directions do not coincide in this model. It has been shown in [5] that the ideal flow condition is that the trajectories of one of the principal stresses in planes of flow coincide with streamlines. It is evident that the ideal flow condition is an additional equation to the standard system of equations of this or that theory of plasticity. However, the proofs of the existence of ideal flows [2, 5, 20] demonstrate that the system of equations comprising the standard constitutive equations and the ideal flow condition is compatible. The difference between the ideal flow solutions and the standard solutions in plasticity lies in boundary conditions. In particular, not all of standard boundary conditions are compatible with the ideal flow condition. In the case of stationary flows, tool surfaces should be frictionless. Moreover, the shape of tool is not fully arbitrary and the design problem is to find such a shape that an ideal flow solution exists.

3. Material model

The double shearing model has been proposed in [16] for soils. Later, it has been shown that this model can be used for metallic materials as well [17]. Under plane strain conditions, the constitutive equations of the model are the Mohr - Coulomb yield criterion and the flow rule. In an arbitrary orthogonal coordinate system (ξ,η) these equations can be written as

$$\left(\sigma_{\xi\xi}+\sigma_{\eta\eta}\right)\sin\phi+\sqrt{\left(\sigma_{\xi\xi}-\sigma_{\eta\eta}\right)^2+4\sigma_{\xi\eta}^2}=2k\cos\phi \qquad (1)$$

and

$$\varepsilon_{\xi\xi}+\varepsilon_{\eta\eta}=0,\quad \sin 2\psi\left(\varepsilon_{\xi\xi}-\varepsilon_{\eta\eta}\right)-2\cos 2\psi\varepsilon_{\xi\eta}-2\sin\phi\left(\omega_{\xi\eta}+d\psi/dt\right)=0. \qquad (2)$$

Here $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\sigma_{\xi\eta}$ are the components of the stress tensor referred to the (ξ,η) coordinate system, $\varepsilon_{\xi\xi}$, $\varepsilon_{\eta\eta}$ and $\varepsilon_{\xi\eta}$ are the components of the strain rate tensor referred to the (ξ,η) coordinate system, $\omega_{\xi\eta}$ is the only non-zero spin (vorticity) component referred to the (ξ,η) coordinate system, ψ is the angle between the $\xi-$ direction and the greatest principal stress measured from the $\xi-$ direction anti-clockwise, d/dt denotes the convected derivative, k is the cohesion, and ϕ is the angle of internal friction. The existence of stationary bulk planar ideal flows has been proven in [5] for the double slip and rotation model proposed in [7]. It is possible to show that this proof is valid for the double shearing model as well. The constitutive equations of the double slip and rotation model are the Mohr - Coulomb yield criterion (1) and the flow rule in the form

$$\varepsilon_{\xi\xi}+\varepsilon_{\eta\eta}=0,\quad \sin 2\psi\left(\varepsilon_{\xi\xi}-\varepsilon_{\eta\eta}\right)-2\cos 2\psi\varepsilon_{\xi\eta}-2\sin\phi\left(\omega_{\xi\eta}+\Omega\right)=0. \qquad (3)$$

Here Ω is the intrinsic spin due to grain rotation. Stationary planar ideal flows for the double slip and rotation model exist if $\Omega=0$ [5]. Consider the second equation in (2) and choose the (ξ,η)

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coordinate systems such that its $\xi-$ coordinate curves are everywhere coincident with streamlines. The ideal flow condition is that the trajectories of one of the principal stresses in planes of flow are everywhere coincident with streamlines. Therefore, $\psi = 0$ or $\psi = \pi/2$ everywhere. In either case, $d\psi/dt = 0$ everywhere. Therefore, the second equation in (2) coincides with the second equation in (3) at $\Omega = 0$ and the models coincide under the ideal flow condition. Hence, the solution given below is valid for both models.

4. System of equations

Let h_ξ and h_η be the scale factors for the $\xi-$ and $\eta-$ coordinate curves, respectively. The coordinate curves of this coordinate system coincide with trajectories of the principal stresses, σ_ξ and σ_η. Assuming that $\sigma_\xi > \sigma_\eta$ it has been shown in [21] that the scale factors satisfy the equation

$$h_\xi h_\eta^b = 1 \tag{4}$$

where $b = (1+\sin\phi)/(1-\sin\phi)$. Any orthogonal net satisfying (4) determines a net of principal stress trajectories giving a solution to the Mohr-Coulomb yield criterion and the equilibrium equations. Then, it has been shown in [5] that the flow rule (3) at $\Omega = 0$ and the ideal flow condition are satisfied if

$$uh_\eta = 1 \tag{5}$$

where u is the magnitude of the velocity vector. According to the ideal flow condition the velocity vector is tangent to the $\xi-$coordinate curves. It is evident that the problem of finding an ideal flow solution is equivalent to the problem of finding a coordinate system satisfying (4). It follows from the geometry of Figure 1 that

$$\frac{\partial x}{\partial \xi} = h_\xi \cos\psi, \quad \frac{\partial x}{\partial \eta} = -h_\eta \sin\psi, \quad \frac{\partial y}{\partial \xi} = h_\xi \sin\psi, \quad \frac{\partial y}{\partial \eta} = h_\eta \cos\psi. \tag{6}$$

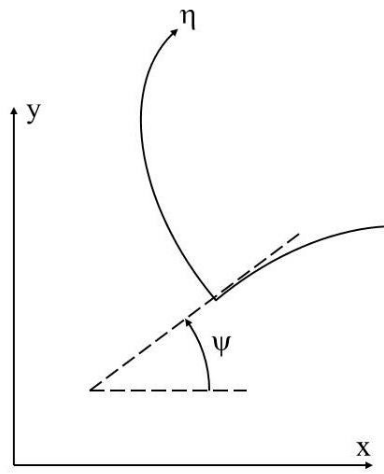

Figure 1. Principal lines and Cartesian coordinates.

The compatibility equations are

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{\partial^2 x}{\partial \eta \partial \xi}, \quad \frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial^2 y}{\partial \eta \partial \xi}. \tag{7}$$

Substituting (6) into (7) and using (4) and (5) yields

$$bu^{b-1}\cos\psi\frac{\partial u}{\partial\eta}-u^b\sin\psi\frac{\partial\psi}{\partial\eta}=\frac{\sin\psi}{u^2}\frac{\partial u}{\partial\xi}-\frac{\cos\psi}{u}\frac{\partial\psi}{\partial\xi},$$
$$bu^{b-1}\sin\psi\frac{\partial u}{\partial\eta}+u^b\cos\psi\frac{\partial\psi}{\partial\eta}=-\frac{\cos\psi}{u^2}\frac{\partial u}{\partial\xi}-\frac{\sin\psi}{u}\frac{\partial\psi}{\partial\xi}. \tag{8}$$

It is always possible to rotate the Cartesian coordinate system so that its *x*- axis is tangent to the $\xi-$ coordinate curve at a given point. In this case $\psi=0$ and equation (8) becomes

$$bu^b\frac{\partial u}{\partial\eta}+\frac{\partial\psi}{\partial\xi}=0,\quad u^{b+2}\frac{\partial\psi}{\partial\eta}+\frac{\partial u}{\partial\xi}=0. \tag{9}$$

Using a standard procedure it is possible to show that this system of equations is hyperbolic with the characteristics

$$\frac{d\eta}{d\xi}=-\sqrt{b}u^{b+1},\quad \frac{d\eta}{d\xi}=\sqrt{b}u^{b+1}. \tag{9}$$

Here the first equation determines the family of $\alpha-$ curves and the second equation the family of $\beta-$ curves. The characteristic relations are

$$\sqrt{b}du-ud\psi=0,\quad \sqrt{b}du+ud\psi=0. \tag{10}$$

Here the first equation is valid on the $\alpha-$ curves and the second equation on the $\beta-$ curves. It follows from (4), (5), and (9) that the angle between the $\xi-$ direction and each of the characteristic directions is $\pi/4+\phi/2$ (Figure 2). Equations (10) can be immediately integrated to give

$$\sqrt{b}\ln u-\psi=-g_1(\beta),\quad \sqrt{b}\ln u+\psi=-g_2(\alpha). \tag{11}$$

Here $g_2(\alpha)$ is an arbitrary function of α and $g_1(\beta)$ is an arbitrary function of β. Solving (11) for *u* and ψ yields

$$2\sqrt{b}\ln u=g_2(\alpha)-g_1(\beta),\quad 2\psi=g_2(\alpha)+g_1(\beta). \tag{12}$$

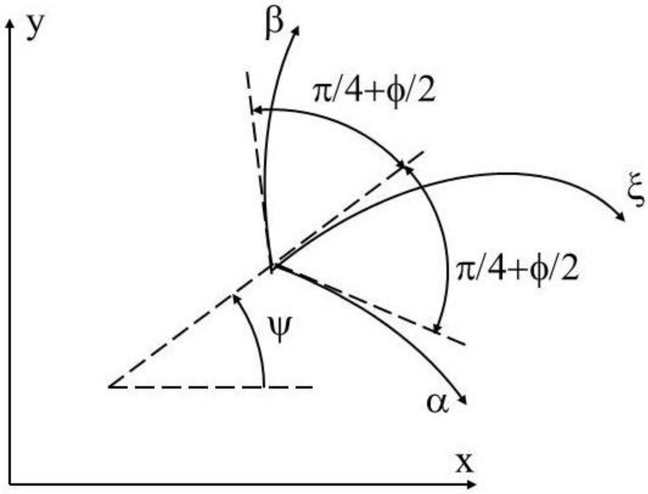

Figure 2. Characteristic coordinates.

In what follows, it is assumed that the characteristic lines of one family are straight. This is an important class of solutions. In particular, the rigid plastic boundary is a straight line if the motion of the rigid zone is a translation. For definitiveness, assume that the $\alpha-$ lines are straight. Let φ be the angle between the x – axis and the $\alpha-$ direction measured from the x – axis anti-clockwise. Then, it is evident that $\psi=\varphi+\pi/4+\phi/2$ and that the angle φ is independent of α. Therefore, the angle ψ is also independent of α and it follows from (12) that $g_2(\alpha)=2g_0$ is constant. Different choices of the function $g_1(\beta)$ merely change the scale of the $\beta-$curves. Therefore, without loss of generality it is possible to choose that $g_1(\beta)=2\beta$. Hence equation (12) becomes

$$\sqrt{b}\ln u=g_0-\beta,\quad \psi=g_0+\beta. \tag{13}$$

5. Ideal flow solution for tool design

A schematic diagram of the process is shown in Figure 3. There are two rigid regions and one plastic region. The motion of each rigid region is a translation. Therefore, the rigid plastic boundaries, *OC* and *AB*, are straight lines. The channel is converging in the direction of flow. Therefore, it follows from the geometry of Figures 2 and 3 that the rigid plastic boundaries are $\alpha-$ lines and equation (13) is valid in the plastic region. The tool surface *OA* and the thickness of the sheet at the exit are prescribed and the tool surface BC that produces an ideal flow should be found from the solution. The origin of the Cartesian coordinate system is chosen at point *O*. The solution can be found by a purely geometric method. The equation of curve *OA* is given in parametric form as

$$x=X(\xi),\quad y=Y(\xi). \tag{14}$$

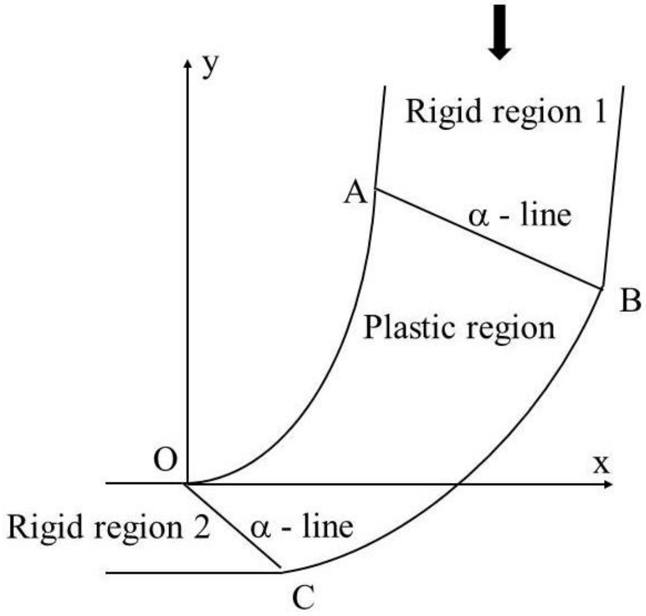

Figure 3. Schematic diagram of the process.

Without loss of generality, it is possible to assume that $\xi=0$ at point *O*. Then,

$$X=0 \quad \text{and} \quad Y=0 \tag{15}$$

at $\xi=0$. Since curve *OA* is an $\xi-$ line, the orientation of the straight $\alpha-$ lines in the plastic region is known (Figure 2). In particular, the equation of these lines is

$$y-Y(\xi)=\left[x-X(\xi)\right]\tan\left(\psi_{OA}-\frac{\pi}{4}-\frac{\phi}{2}\right) \tag{16}$$

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where ψ_{OA} is the value of ψ on curve *OA*. Using trigonometric relations equation (16) can be transformed to

$$y - Y(\xi) = \left[x - X(\xi)\right]\left(\frac{\tan\psi_{OA} - m}{1 + m\tan\psi_{OA}}\right) \tag{17}$$

where $m = \tan(\pi/4 + \phi/2)$. Using (14) it is possible to find that

$$\tan\psi_{OA} = \left(\frac{dY}{d\xi}\right)\left(\frac{dX}{d\xi}\right)^{-1} = \lambda(\xi). \tag{18}$$

The curve *BC* is also an $\xi-$ line. Therefore, the equation of this curve is

$$\frac{dy}{dx} = \tan\psi_{OA} = \lambda(\xi). \tag{19}$$

Differentiating (17) with respect to ξ and using (18) leads to

$$\frac{dy}{d\xi} - \frac{dY}{d\xi} = \left(\frac{dx}{d\xi} - \frac{dX}{d\xi}\right)\left(\frac{\lambda - m}{1 + m\lambda}\right) + (x - X)\Lambda \tag{20}$$

where

$$\Lambda(\xi) = \frac{d\lambda}{d\xi}\frac{(1 + m^2)}{(1 + m\lambda)^2}. \tag{21}$$

Equation (19) can be rewritten as

$$\frac{dy}{d\xi} = \lambda\frac{dx}{d\xi}. \tag{22}$$

Eliminating the derivative $dy/d\xi$ in (20) by means of (22) yields

$$m\frac{dx}{d\xi} - x\Lambda\left(\frac{1 + m\lambda}{\lambda^2 + 1}\right) = \left(\frac{dY}{d\xi} - X\Lambda\right)\left(\frac{1 + m\lambda}{\lambda^2 + 1}\right) - \left(\frac{\lambda - m}{\lambda^2 + 1}\right)\frac{dX}{d\xi}. \tag{23}$$

This is a linear differential equation for *x*. Its general solution can be found without any difficulty. Using (19) equation (17) can be rewritten as

$$x - X = (y - Y)\left(\frac{1 + m\lambda}{\lambda - m}\right) \tag{24}$$

Differentiating this equation with respect to ξ leads to

$$\frac{dx}{d\xi} - \frac{dX}{d\xi} = \left(\frac{dy}{d\xi} - \frac{dY}{d\xi}\right)\left(\frac{1 + m\lambda}{\lambda - m}\right) + (y - Y)P \tag{25}$$

where

$$P(\xi) = -\frac{d\lambda}{d\xi}\frac{(1 + m^2)}{(\lambda - m)^2}. \tag{26}$$

Eliminating the derivative $dx/d\xi$ in (25) by means of (22) yields

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$$m\frac{dy}{d\xi}-yP\lambda\left(\frac{m-\lambda}{\lambda^2+1}\right)=\lambda\left(\frac{dX}{d\xi}-YP\right)\left(\frac{m-\lambda}{\lambda^2+1}\right)+\lambda\left(\frac{1+m\lambda}{\lambda^2+1}\right)\frac{dY}{d\xi}. \tag{27}$$

This is a linear differential equation for y. The solution of equations (23) and (27) determines the shape of BC. The thickness of the sheet at the exit is denoted by H. Therefore, the boundary conditions to equations (23) and (27) are

$$x=\frac{H\sin\left(\pi/4-\phi/2+\psi_0\right)}{\cos\left(\pi/4-\phi/2\right)} \quad \text{and} \quad y=-\frac{H\cos\left(\pi/4-\phi/2+\psi_0\right)}{\cos\left(\pi/4-\phi/2\right)} \tag{28}$$

at $\xi=0$, respectively. Here ψ_0 is the value of ψ_{OA} at $\xi=0$.

6. Illustrative example

Assume that curve OA is a parabola (Figure 3). Taking into account (14) and (15) its equation can be written as

$$X(\xi)=\xi, \quad Y(\xi)=n\xi^2 \tag{29}$$

where n is constant. Then, it follows from (18), (21) and (26) that

$$\lambda=2n\xi, \quad \Lambda=\frac{2n\left(1+m^2\right)}{\left(1+2mn\xi\right)^2}, \quad P=-\frac{2n\left(1+m^2\right)}{\left(2n\xi-m\right)^2}. \tag{30}$$

Substituting (29) and (30) into (23) yields

$$\frac{dx}{d\xi}-\frac{2n\left(1+m^2\right)x}{m\left(4n^2\xi^2+1\right)\left(1+2nm\xi\right)}=1+\frac{1}{\left(1+2mn\xi\right)}-\frac{\left(m+2n\xi\right)}{m\left(1+4n^2\xi^2\right)}. \tag{31}$$

It follows from (30) that $\psi_0=0$. The solution of equation (31) satisfying the boundary condition (28) at $\psi_0=0$ is

$$x=\frac{\left(1+2mn\xi\right)}{\sqrt{1+4n^2\xi^2}}\exp\left[\frac{\arctan\left(2n\xi\right)}{m}\right]\times$$
$$\left\{H\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+\int_0^{\xi}\frac{\sqrt{1+4n^2z^2}}{\left(1+2mnz\right)}\exp\left[-\frac{\arctan\left(2nz\right)}{m}\right]\left[1+\frac{1}{\left(1+2mnz\right)}-\frac{\left(m+2nz\right)}{m\left(1+4n^2z^2\right)}\right]dz\right\}. \tag{32}$$

Here z is a dummy variable of integration. Substituting (29) and (30) into (27) yields

$$\frac{dy}{d\xi}+\frac{4n^2\left(1+m^2\right)\xi y}{m\left(4n^2\xi^2+1\right)\left(m-2n\xi\right)}=\frac{2n\xi\left[m^2-2nm\xi+2\left(1+3m^2\right)n^2\xi^2-8mn^3\xi^3\right]}{m\left(1+4n^2\xi^2\right)\left(m-2n\xi\right)}. \tag{33}$$

The solution of this equation satisfying the boundary condition (28) at $\psi_0=0$ is

$$y=\frac{2n}{m}\frac{\sqrt{1+4n^2\xi^2}}{\left(m-2n\xi\right)}\exp\left[-\frac{\arctan\left(2n\xi\right)}{m}\right]\times$$
$$\left\{\int_0^{\xi}z\exp\left[\frac{\arctan\left(2nz\right)}{m}\right]\frac{\left[m^2-2nmz+2\left(1+3m^2\right)n^2z^2-8mn^3z^3\right]}{\left(1+4n^2z^2\right)^{3/2}}dz-\frac{Hm}{2n}\right\}. \tag{34}$$

Equations (32) and (34) determine curve BC in parametric form with ξ being the parameter.

7. Conclusions
The theory of planar bulk ideal flows for tool design has been developed assuming that one family of characteristic curves is straight. The theory is valid for both the double – shearing model [16] and the double slip and rotation model [7] of pressure-dependent plasticity. Finding an optimal tool shape has been reduced to solving linear ordinary differential equations. An illustrative example has been given assuming that one surface of tool is determined by equation (29) and that the motion of each rigid region is a translation. Then, the other surface is determined from the solution and is given by equations (32) and (34). It is seen from these equations that the solution is practically analytic. A numerical technique is only necessary to evaluate ordinary integrals.

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