

Differential quadrature method of nonlinear bending of functionally graded beam

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Abstract. Using the third-order shear deflection beam theory (TBT), nonlinear bending of functionally graded (FG) beams composed with various amounts of ceramic and metal is analyzed utilizing the differential quadrature method (DQM). The properties of beam material are supposed to accord with the power law index along to thickness. First, according to the principle of stationary potential energy, the partial differential control formulae of the FG beams subjected to a distributed lateral force are derived. To obtain numerical results of the nonlinear bending, non-dimensional boundary conditions and control formulae are dispersed by applying the DQM. To verify the present solution, several examples are analyzed for nonlinear bending of homogeneous beams with various edges. A minute parametric research is in progress about the effect of the law index, transverse shear deformation, distributed lateral force and boundary conditions.

1. Introduction

It is very clear that analysis of the nonlinear bending for FGM beams under various shear deflection theories consists of two accented phases, for example, building the mathematical pattern. These problems are represented by a crowd of the partial differential equations, either nonlinear or linear, it is difficult to establish the analytical solutions. Therefore, the approximate numerical solution is adopted to meet the requirements of the project. At present, the focus of a large amount of literature is on FGM beams under the mechanical load.

For exact solutions, Sankar [1] obtained an exact elasticity solution for FG beams that considers the static transverse loading; this was later adopted for getting analytical solutions regarding stresses about a sandwich beam. Yu and Zhong [2] obtained a general solution for a beam with the uniform loading on the upper surface by using a semi-inverse method. Huang [3] et al. studied the exact solutions for anisotropic FG material beams with the different boundary conditions. Based on the shear deformation theory, Ma and Lee [4] gave a analytical solution about nonlinear responses of a FG beam. Lü et al. [5] obtained a exact solution for the bending of FGM beams to adopt the differential quadrature method.

For numerical solutions, using TBT, the shear, axial stresses and the transverse deflection in the thick FGM bar under an uniform loading for the various boundary conditions are studied in detailed [6]. By using the mathematical similarities among the governing equations, Li et al. [7] derived the analytical relationship of bending solutions. Li [8] obtained a new way to study the static responses of classic and first-order beams. Application of TBT and physical neutral surface, Zhang and Zhou [9] built a mathematical model of beams to obtain the numerical results of bending by the Multi-term Ritz method. Under mechanical loading, Niknam et al. [10] studied nonlinear static responses of the tapered beam under the various conditions. For the static and dynamic analysis of beams subjected to thermal load, Ma and Lee [11] carried out the related research by a shooting method. In this paper,



based on TBT, a numerical result by DQ method is obtained about nonlinear bending of FG beams subjected to mechanical loading under C-C and S-S boundary conditions.

2. Mathematical problem

2.1. Model material properties

Take into account a FGM beam (width b , length l , and height h) with a rectangular section (Figure 1), which is mainly made up of ceramics and metals. The material properties vary continuously along the height direction from the pure ceramic surfaces to the pure metal surfaces. Young's modulus E and the passion ratio ν are expressed as following:

$$P(z) = P_C + (P_M - P_C)V_M \quad (1)$$

where M and C denote the components of metal and ceramic; and V_M is the percentage of the volume of metal components. This formula changes according to below function:

$$V_M = \left(\frac{1}{2} - \frac{z}{h} \right)^n \quad (2)$$

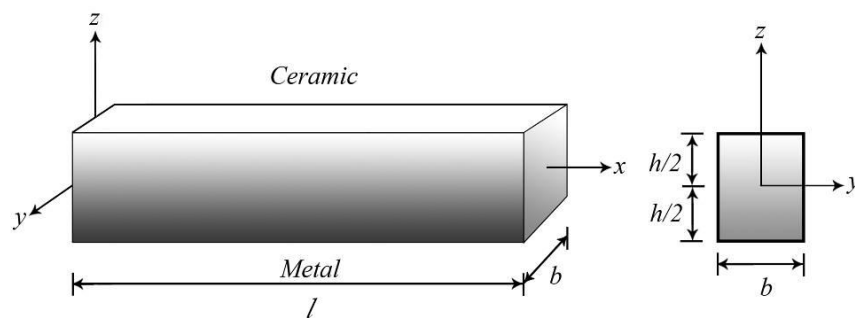


Figure 1. Geometry model of the studied FG beam.

2.2. The equilibrium equations

Displacement field as following:

$$\begin{cases} U_x(x, z) = \phi(x) + u(x) - \beta z^3 \phi(x) - \beta z^3 w_{,x}(x) \\ U_z(x, z) = w(x) \end{cases} \quad (3)$$

where, x is the horizontal ordinate along the longitudinal direction of model beam; $U_x(x, z)$ and $U_z(x, z)$ are the deformation of x , z axial at arbitrary points of the studied beam; $w(x)$ denotes the vertical deflection; $\phi(x)$ expresses as the rotational angle.

Formula expression of displacement-strain are quoted by Heyliger and Reddy[12]:

$$\begin{cases} \epsilon_x = z\phi_{,x} + \frac{w_{,x}^2}{2} + u_{,x} - \beta z^3(\phi_{,x} + w_{,xx}) \\ \gamma_{xz} = w_{,x} + \phi - 3\beta z^2(\phi + w_{,x}) \end{cases} \quad (4)$$

where $\beta = \frac{4}{3h^2}$

As a result of the materials of the FG beam obey Hooke's law, we find the following stress-displacement relations:

$$\sigma_x = E(z)\epsilon_x \quad (5)$$

$$\tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} \quad (6)$$

2.3. Third-order shear deformation theory (TBT)

Based on TBT, The equilibrium equations can be expressed by:

$$\begin{cases} \frac{d^2 u}{dx^2} + \frac{dw}{dx} \frac{d^2 w}{dx^2} - \frac{\beta E_{11} \Omega_2}{\Omega_1} \frac{d^3 w}{dx^3} - \frac{\beta E_{11} \Omega_3}{A_{11}} \varphi = 0 \\ \frac{d^2 \varphi}{dx^2} - \frac{A_{11} \Omega_2}{\Omega_1} \frac{d^3 w}{dx^3} - \Omega_3 \varphi = 0 \\ \left(\frac{\beta^2 E_{11}^2 \Omega_2^2}{\Omega_1} - \beta F_{11} + \beta \bar{F}_{11} \Omega_2 \right) \frac{d^4 w}{dx^4} + \Omega_2 \Omega_3 \tilde{D}_{11} \frac{d\varphi}{dx} + \left\{ A_{11} \left[\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right] - \beta E_{11} \frac{d\varphi}{dx} \right\} \frac{d^2 w}{dx^2} + q = 0 \end{cases} \quad (7)$$

Where

$$\Omega_1 = A_{11} - \frac{\beta^2 E_{11}^2}{\tilde{D}_{11}}, \Omega_2 = 1 + \frac{\beta \bar{F}_{11}}{\tilde{D}_{11}}, \Omega_3 = \frac{A_{11} \tilde{A}_{44}}{\Omega_1 \tilde{D}_{11}}, \varphi = \phi + \frac{dw}{dx}, \tilde{A}_{44} = \bar{A}_{44} - 3\beta \bar{D}_{44}, \bar{A}_{44} = A_{44} - 3\beta D_{44}, \\ \tilde{D}_{11} = \bar{D}_{11} - \beta \bar{F}_{11}, \bar{D}_{11} = D_{11} - \beta F_{11}.$$

According to the above equations, the non-dimension quantities are defined by:

$$\xi = \frac{x}{l}, W = \frac{w}{h}, \bar{\varphi} = \frac{\phi l}{h}, U = \frac{ul}{h^2}, \beta_0 = \frac{l}{h}, p = \frac{Pl^2}{D_x}, S_{T0} = \frac{\beta^2 E_{11}^2 \Omega_2^2}{\Omega_1} - \beta F_{11} + \beta \bar{F}_{11} \Omega_2, S_{T1} = \frac{\beta E_{11} \Omega_2}{\Omega_1 h}, \\ S_{T2} = \frac{\beta E_{11} \Omega_3 l^2}{A_{11} h}, S_{T3} = \frac{A_{11} \Omega_2}{\Omega_1}, S_{T4} = \Omega_3 l^2, S_{T5} = \frac{\Omega_2 \Omega_3 \tilde{D}_{11} l^2}{S_{T0}}, S_{T6} = \frac{A_{11} h^2}{S_{T0}}, q_{T0} = \frac{ql^4}{h S_{T0}}.$$

From Eq. (7), we obtain the non-dimension equilibrium equations as following:

$$\begin{cases} \frac{d^2 U}{d\xi^2} + \frac{dW}{d\xi} \frac{d^2 W}{d\xi^2} - S_{T1} \frac{d^3 W}{d\xi^3} - S_{T2} \bar{\varphi} = 0 \\ \frac{d^2 \bar{\varphi}}{d\xi^2} - S_{T3} \frac{d^3 W}{d\xi^3} - S_{T4} \bar{\varphi} = 0 \\ \left(\frac{d^4 W}{d\xi^4} + S_{T5} \frac{d\bar{\varphi}}{d\xi} + S_{T6} \left\{ \left[\frac{dU}{d\xi} + \frac{1}{2} \left(\frac{dW}{d\xi} \right)^2 \right] - \frac{S_{T2}}{S_{T4}} \frac{d\bar{\varphi}}{d\xi} \right\} \frac{d^2 W}{d\xi^2} + q_{T0} = 0 \end{cases} \quad (8)$$

3. Mathematical formulation

3.1. The weighting coefficients

The DQM must discretize the definition domain into m points. Along the definition domain, a weighted nonlinear summation of all of the functional values in m points approximately expresses the derivatives of any point [13-14].

$$\frac{d^k f(x_i)}{dx^k} = f_i^{(k)} = \sum_{j=1}^m A_{ij}^{(k)} f_j, \quad i = 1, 2, \dots, m \quad (9)$$

where m denotes the number of grid points, $A_{ij}^{(k)}$ is the weighting coefficient for the k th derivative.

3.2. Selection of the sampling nodes

The sampling points of differential quadrature representations are selected as shifted Chebyshev-Gauss-Lobatto points, which are given as follows:

$$x_i = \frac{1}{2} \left\{ 1 - \cos \left[\frac{\pi(i-1)}{m-1} \right] \right\}, \quad i = 1, 2, \dots, m \quad (10)$$

4. Numerical results and discussions

For numerical analysis, we calculate an Si₃N₄-SUS304 FG beam under a uniformly distributed loading $q(\text{KNcm}^{-1})$, with a length of $l=100\text{cm}$, a height of $h=1\text{cm}$, a width of $b=1\text{cm}$, and the number of nodes is $m=11$. The material properties: $E_{\text{Si}_3\text{N}_4}=348.43\text{GPa}$, $E_{\text{SUS304}}=201.04\text{GPa}$, $\nu_{\text{SUS304}}=0.24$, $\nu_{\text{Si}_3\text{N}_4}=0.33$.

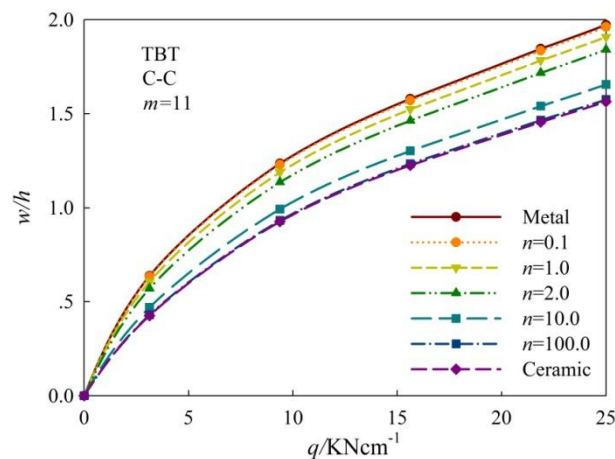


Figure 2. Central deflection-loading curves of a C-C FG beam.

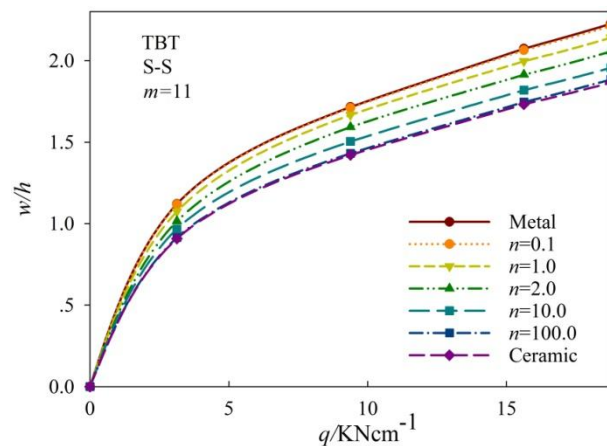


Figure 3. Central deflection-loading curves of a S-S FG beam.

From figures 2 to 3, it can be observed that the shear deformation effects is the critical factor to cause an increase of the central deflection. For an Si₃N₄-SUS304 FG beam with the C-C and S-S boundary conditions, using a shear deformation beam theory, dropping the power law index will cut down the stiffness of FG beam, in succession, results in a decrease of the deflections.

5. Conclusions

In this article, based on TBT, the nonlinear bending of FG beams exerted under two different edges is analytically investigated. Nonlinear curves are gradually obvious with the increasing of load. Due to consider shear deformation, the central deflection of an FG beam using TBT is greater compared to homogeneous beam. The central deflection-load curves obtained by exerting an simply supported edge is faster, and gradually becomes smoothly with load increase. Using TBT, the maximum central deflections of Si₃N₄-SUS304 beams is not significantly different from the center deflections found using FBT. In addition, dropping the power law index will cut down the stiffness of FG beam, in succession, results in a decrease of the deflections.

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