Generalized Moore Penrose Inverse of Normal Elements in a Ring with Involution

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Abstract. Based on the definition of a normal element in a ring with involution, it is found that each normal element is commutatively with the product of itself and the involution of itself. On the other hand, if the element of a ring with involution has generalized Moore Penrose inverse, then the element is also commutative with the product of itself and the involution of itself. In this paper, the phenomenon of the similarity properties from normal elements and generalized Moore Penrose inverse is used to establish the relationship between of them with them. .

1. Introduction

Let R be a ring with identity and $a \in R$. An involution "$*$" in a ring R is unary operation $a \in R \mapsto a^* \in R$ such that [1, 3]

$$(a^*)^* = a \qquad (a+b)^* = a^* + b^* \qquad (ab)^* = b^* a^*$$

for each elements a, b in a ring R.

In ring R with involution "$*$", element $a \in R$ is called normal if $aa^* = a^* a$. It is an extension of the normal matrix definition in the set of matrices over real number [4] . The set of normal elements in R is symbolized by R^{nor}.

Cvetkovi´c, et.al. [3] explain that the group inverse of $a \in R$ is an element $a^{\#} \in R$ such that :

$$aa^{\#}a = a, \qquad a^{\#}aa^{\#} = a^{\#}, \qquad aa^{\#} = a^{\#}a.$$

The set of all group invertible elements of R will be denoted by $R^{\#}$.

Koliha and Patrıcio [2] say that an element $a \in R$ is *-cancellable if for any $x \in R$ the following hold :

$$a^*ax = 0 \Rightarrow ax = 0 \text{ and } xaa^* = 0 \Rightarrow xa = 0.$$

Suppose R is a ring with involution "$*$". Element $b \in R$ is generalized Moore Penrose inverse of $a \in R$ if the followings are satisfied [5] :

$$aba = a, \qquad (ab)^* = ab, \qquad (ba)^* = ba. \tag{1}$$

Any b that hold (1) is a generalized Moore–Penrose inverse of a. We use a_g^+ and R_g^+, respectively to denote generalized Moore–Penrose inverse of a and the set of generalized Moore Penrose invertible elements in R.

If $a \in R$ is the generalized Moore Penrose invertible element, then a_g^+ is not usually unique. That are $a_g^+ = a_g^+ a a_g^+$ or $a_g^+ \neq a_g^+ a a_g^+$. The set of generalized Moore Penrose invers of $a \in R$ will be denoted by $(a_g^+)^-$.
The next theorem describes the existence of the generalized Moore Penrose inverse on the ring with involution that has been given in [5, Theorem 2.10].

Theorem 1: *For any $a \in R$ the following equivalent conditions holds* :
(1) $a \in R_g^+$,
(2) $a^* \in R_g^+$,
(3) *a is *-cancellable and* $a^*a \in R_g^+$,
(4) *a is *-cancellable and* $aa^* \in R_g^+$,
(5) *a is *-cancellable and* $a^*a \in R^\#$,
(6) *a is *-cancellable and* $aa^* \in R^\#$,
(7) *a is *-cancellable and both a^*a and aa^* are regular,*
(8) $a \in a^*R \cap Ra^*a$,
(9) *a is *-cancellable and a^*aa^* regular*

Theorem 2 in the following describes the properties of a_g^+ if $a \in R_g^+$, which has been discussed in [5, Theorem 2.5].

Theorem 2: *If $a \in R_g^+$, then*

1. $(a^*)_g^+ a^* (a^*)_g^+ = (a_g^+ a a_g^+)^*$ for each $a_g^+ \in (a_g^+)^-$ and $(a^*)_g^+ \in ((a^*)_g^+)^-$.
2. $(a^*a)_g^+ a^* a (a^*a)_g^+ = a_g^+ (a_g^+ a a_g^+)^*$ and $(aa^*)_g^+ aa^* (aa^*)_g^+ = (a_g^+ a a_g^+)^* a_g^+$ for each $a_g^+ \in (a_g^+)^-$, $(aa^*)_g^+ \in ((aa^*)_g^+)^-$ and $(a^*a)_g^+ \in ((a^*a)_g^+)^-$.
3. $a^* = a^* a a_g^+ = a_g^+ a a^*$ for each $a_g^+ \in (a_g^+)^-$.
4. $a_g^+ a a_g^+ = (a^*a)_g^+ a^* a (a^*a)_g^+ a^* = a^* (aa^*)_g^+ aa^* (aa^*)_g^+$ for each $a_g^+ \in (a_g^+)^-$, $(aa^*)_g^+ \in ((aa^*)_g^+)^-$ and $(a^*a)_g^+ \in ((a^*a)_g^+)^-$.
5. $(a^*)_g^+ a^* (a^*)_g^+ = a(a^*a)_g^+ a^* a (a^*a)_g^+ = (aa^*)_g^+ aa^* (aa^*)_g^+ a$ for each $(a^*)_g^+ \in ((a^*)_g^+)^-$, $(aa^*)_g^+ \in ((aa^*)_g^+)^-$ and $(a^*a)_g^+ \in ((a^*a)_g^+)^-$.
6. $(a_g^+ a a_g^+)_g^+ = a$ for each $a_g^+ \in (a_g^+)^-$.

By definition of normal element, if $a \in R^{nor}$ then

$$aa^*a = a^*aa \tag{2}$$

and

$$aaa^* = aa^*a \tag{3}$$

Equation (2) and (3) show that a is commutative with a^*a and aa^*. According to Theorem 1, if $a \in R_g^+$ then $aa^*, a^*a \in R^\#$. Udjiani, et.al. [5] have observed that

$$(aa^*)^\# = (a\,a^*)_g^+ a\,a^*(a\,a^*)_g^+ \quad (4)$$

and

$$(a^*a)^\# = (a^*a)_g^+ a^*a(a^*a)_g^+ \quad (5)$$

for each $(aa^*)_g^+ \in \left((aa^*)_g^+\right)^-$ and $(a^*a)_g^+ \in \left((a^*a)_g^+\right)^-$. (Existence of $(a^*a)_g^+$ guaranted by Theorem 1)

In other side, Mosic dan Djordjevic [4] say that if $a \in R$ is group invertible then a is double commutative with $a^\#$. So, if $a \in R_g^+$ then aa^* is double commutative with $(aa^*)^\#$ and a^*a is double commutative with $(a^*a)^\#$. From the explanation we can conclude that there are a relation between normal element and generalized Moore Penrose invertible element. Motivated by Mosic dan Djordjevic [4], this paper discusses the properties of elements in $R_g^+ \cap R^{nor}$.

2. Normal Elements and generalized Moore Penrose invertible element.

The In this section normal elements in rings with involution are characterized by conditions involving their group and Generalized Moore–Penrose inverse. Some of these results are proved for set of matrices over real number with transpose involution.

From Theorem 1, if $a \in R_g^+$ then a^*a, $a\,a^* \in R^\#$. Inspired that if $a \in R$ is group invertible then a is double commutative with $a^\#$, so we get Lemma 3 in the following :

Lemma 3: *If $aa^* \in R^\#$, $x \in R$ and $aa^*x = xaa^*$ then $(aa^*)^\#x = x(aa^*)^\#$. Also if $a^*a \in R^\#$, $x \in R$ and $a^*ax = x\,a^*a$ then $(a^*a)^\#x = x(a^*a)^\#$.*

Proof :

$$\begin{aligned}(a\,a^*)^\#x &= ((a\,a^*)^\#)^2 a\,a^*x = ((a\,a^*)^\#)^2 xa\,a^* = ((a\,a^*)^\#)^2 x\,(a\,a^*)^2(a\,a^*)^\# \\ &= ((a\,a^*)^\#)^2 a\,a^*\,xa\,a^*(a\,a^*)^\# = (a\,a^*)^\#\,xa\,a^*(a\,a^*)^\# \\ &= (a\,a^*)^\#\,a\,a^*x(a\,a^*)^\# = (a\,a^*)^\#\,a\,a^*xa\,a^*((a\,a^*)^\#)^2 \\ &= (a\,a^*)^\#\,a\,a^*a\,a^*x((a\,a^*)^\#)^2 = a\,a^*x((a\,a^*)^\#)^2 = xa\,a^*((a\,a^*)^\#)^2 \\ &= x(a\,a^*)^\#.\end{aligned}$$

By the same way we have $a^*a \in R^\#$, $x \in R$ and $a^*ax = x\,a^*a$ then $(a^*a)^\#x = x(a^*a)^\#$. ■

The next Theorem 4 is inspired by the work of Lemma 1.1. in [4].

Theorem 4: *Suppose $a \in R_g^+$ and $b \in R$. If $ab = ba$ and $a^*b = ba^*$ then $a_g^+aa_g^+b = b\,a_g^+aa_g^+$ for each $a_g^+ \in (a_g^+)^-$.*

Proof :

Using Theorem 2, if $a \in R_g^+$ then $a_g^+aa_g^+ = a^*(aa^*)_g^+aa^*(aa^*)_g^+$ for each $a_g^+ \in (a_g^+)^-$ and $(aa^*)_g^+ \in \left((aa^*)_g^+\right)^-$.

Futhermore, using Equation (4) we have

$$a_g^+aa_g^+ = a^*(a\,a^*)^\# \text{ for each } a_g^+ \in (a_g^+)^- \quad (6)$$

By Lemma [3], we have $(a\,a^*)^\#$ is double commutative with $a\,a^*$. Since b is commutative with a^*, then $(a\,a^*)^\#$ is commutative with *b*.
By Equation (6), we obtain $b\,a_g^+ aa_g^+ = b\,a^*(a\,a^*)^\# = a^* b(a\,a^*)^\# = a^*(a\,a^*)^\# b = a_g^+ aa_g^+ b$ for each $a_g^+ \in (a_g^+)^-$. ∎

Normal and generalized Moore Penrose invertible element have the same properties, that is commutative with *a* and a^*. Motivated by Mosic dan Djordjevic [4, Lemma 1.2.], we can construct the existence of normal element in R_g^+ in Theorem 5 as follows :

Theorem 5: *Let* $a \in R_g^+$. *Element* a *is normal if and only if* $aa_g^+ = a_g^+ a$ *and* $a^* a_g^+ aa_g^+ = a_g^+ aa_g^+ a^*$ *for each* $a_g^+ \in (a_g^+)^-$.
Proof :
By properties of normal element, if $a \in R^{nor}$ then *a* commutative with aa^* and $a^* a$. In other side if $a \in R_g^+$, then by Theorem 1 we have aa^*, $a^* a \in R^\#$. So, using Lemma 3, we conclude $aa_g^+ (a_g^+ aa_g^+)^* = a_g^+ (a_g^+ aa_g^+)^* a$ for each $a_g^+ \in (a_g^+)^-$.
Futhermore $aa_g^+ = aa_g^+ aa_g^+ = aa_g^+ (aa_g^+)^* = aa_g^+ (a_g^+ aa_g^+)^* a^* = a_g^+ (a_g^+ aa_g^+)^* aa^* = a_g^+ (a_g^+ aa_g^+)^* a^* a = a_g^+ a$ for each $a_g^+ \in (a_g^+)^-$. In the same way $a^* a_g^+ aa_g^+ = a^* a_g^+ (aa_g^+)^* = a^* a_g^+ (a_g^+ aa_g^+)^* a^* = a_g^+ (a_g^+ aa_g^+)^* a^* a^* = a_g^+ aa_g^+ a^*$ for each $a_g^+ \in (a_g^+)^-$. Conversely if $aa_g^+ = a_g^+ a$ and $a^* a_g^+ aa_g^+ = a_g^+ aa_g^+ a^*$ for each $a_g^+ \in (a_g^+)^-$, then using Theorem 2 we have $aa^* = aa^* aa_g^+ = aa^* a_g^+ a = aa^* a_g^+ aa_g^+ a = aa_g^+ aa_g^+ a^* a = a^* a$. ∎

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