IOP Conf. Series: Materials Science and Engineering **300** (2018) 012072 doi:10.1088/1757-899X/300/1/012072

Notes on matrix of Fibonacci numbers

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Abstract. For integer $m \geq 3$, we discuss m by m matrices of Fibonacci numbers that resemble the simplicity of Binet formula and maintain the calculation using only integer computation. We show that the m by m matrices of Fibonacci numbers will eventually store $m + 1$ distinct consecutive Fibonacci numbers.

1. Introduction

One of the most well known sequence is the Fibonacci sequence (Fibonacci numbers). The Fibonacci sequence $\{f_n\}_{n\geq 0}$ is the sequence of numbers of the form

$$\begin{array}{ccccccccc} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & \dots \\ 0 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \dots \end{array}$$

Interesting problem regarding Fibonacci numbers is to find the n^{th} term of the sequences. Around 1875 Binet proposed a formula that can calculate the n^{th} term Fibonacci numbers known as Binet formula [1, 4]. For $n \geq 2$, Binet formula can be formulated as follows

$$f_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{1}$$

with $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$. One difficulty in determining the n^{th} term f_n of Fibonacci numbers is the fact that the formula (1) dealing with calculation of irrational number.

One way to find the Fibonacci numbers, without dealing with irrational number, is using a recurrence relation that based on the property that for $n \geq 2$ each term the Fibonacci numbers is the sum of the previous two terms [4]. So the Fibonacci numbers can be found using recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \quad \text{for} \quad n \geq 2. \tag{2}$$

We note that the recurrence relation (2) will store three consecutive terms of the Fibonacci numbers.

Based on the recurrence relation (2) an attempt has been made to store three terms of Fibonacci numbers using a 2 by 2 matrix [2–4]. In this paper, for $m \geq 3$ we establish an m by m matrix that will eventually store $m + 1$ terms of the Fibonacci numbers. We organize the paper as follows. In Section 2, we review the 2 by 2 matrix for Fibonacci numbers. In Section 3, based on the 2 by 2 matrix of Fibonacci numbers, for $m \geq 3$ we discuss an m by m matrix of Fibonacci numbers.

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2. Case of 2 by 2 matrix

We review the 2 by 2 matrix of Fibonacci numbers as in [2–4]. We start with the special 2 by 2 matrix $F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Let $U_n = \begin{pmatrix} u_{n,1} \\ u_{n,2} \end{pmatrix}$ and define a recurrence relation

$$U_{n+1} = \begin{pmatrix} u_{n+1,1} \\ u_{n+1,2} \end{pmatrix} := F_2 U_n = \begin{pmatrix} u_{n,1} + u_{n,2} \\ u_{n,1} \end{pmatrix}.$$

Notice that $u_{n+1,1} = u_{n,1} + u_{n,2}$ resembles a Fibonacci like recurrences as in (2). Hence if we choose the vector U_n to be $U_n = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$ for some two consecutive terms of Fibonacci numbers, then the vector U_{n+1} will store the two consecutive terms f_{n+1} and f_n of the Fibonacci numbers.

Fortunately, the entries of each column of

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix} \tag{3}$$

are two consecutive Fibonacci numbers. This implies the power of F_2 for some positive integer $n \geq 1$ will store three consecutive Fibonacci numbers as stated in the following theorem.

Theorem 1. [4] *Let $\{f_n\}_{n\geq 0}$ be the squence of Fibonacci numbers. If $F_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then $(F_2)^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$ for $n \geq 1$.*

Proof. We prove by mathematical induction on n. If $n = 1$, we have $(F_2)^1 = \begin{pmatrix} f_{1+1} & f_1 \\ f_1 & f_{1-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ as in (3).

Assume that for some positive integer $k \geq 1$, $(F_2)^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$. This implies

$$\begin{aligned} (F_2)^{k+1} &= (F_2)(F_2)^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} f_{k+1} + f_k & f_k + f_{k-1} \\ f_{k+1} & f_k \end{pmatrix} = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix} \\ &= \begin{pmatrix} f_{(k+1)+1} & f_{(k+1)} \\ f_{(k+1)} & f_{(k+1)-1} \end{pmatrix}. \end{aligned}$$

Therefore, we conclude that $(F_2)^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$ for $n \geq 1$. □

We note that the statement of Theorem 1 constitutes the beauty of Binet formula (1) of finding Fibonacci numbers using a simple formula and the beauty of the recurrence relation (2) that finds the Fibonacci numbers using only integer computation.

As a direct consequence of Theorem 1 we have the following result.

Corollary 2. *Let $\{f_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers. If $F = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix}$ for some positive integer k, then $F^n = \begin{pmatrix} f_{(k+1)n+1} & f_{(k+1)n} \\ f_{(k+1)n} & f_{(k+1)n-1} \end{pmatrix}$.*

Proof. Notice that $F = \begin{pmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1}$. Hence,

$$F^n = \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1}\right]^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{(k+1)n} = \begin{pmatrix} f_{(k+1)n+1} & f_{(k+1)n} \\ f_{(k+1)n} & f_{(k+1)n-1} \end{pmatrix}.$$

□

3. Our results

We will generalize the result in Theorem 1 for m by m matrix for some $m \geq 3$. We start with $m = 3$. Let $F_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 & 0 \\ f_1 & f_0 & 0 \\ f_0 & 1 & 0 \end{pmatrix}$. Let U_{n+1} and U_n be 3 by 1 matrices and consider the recurrence

$$\begin{pmatrix} u_{n+1,1} \\ u_{n+1,2} \\ u_{n+1,3} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_{n,1} \\ u_{n,2} \\ u_{n,3} \end{pmatrix}.$$

Then $u_{n+1,1} = u_{n,1} + u_{n,2}$ and $u_{n+1,j} = u_{n,j-1}$ for $j = 2, 3$. As in the case of 2 by 2 matrix, if we replace U_n with vector with entries consist of three consecutive Fibonacci numbers then FU_n will result in three consecutive Fibonacci numbers.

Proposition 3. *Let $\{f_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers. If* $F_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, *then*

$(F_3)^n = \begin{pmatrix} f_{n+1} & f_n & 0 \\ f_n & f_{n-1} & 0 \\ f_{n-1} & f_{n-2} & 0 \end{pmatrix}$ *for $n \geq 2$.*

Proof. We proof by induction on n. If $n = 2$, we have

$$\begin{aligned} (F_3)^2 &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} f_3 & f_2 & 0 \\ f_2 & f_1 & 0 \\ f_1 & f_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f_{2+1} & f_2 & 0 \\ f_2 & f_{2-1} & 0 \\ f_{2-1} & f_{2-2} & 0 \end{pmatrix}. \end{aligned}$$

Assume that $(F_3)^k = \begin{pmatrix} f_{k+1} & f_k & 0 \\ f_k & f_{k-1} & 0 \\ f_{k-1} & f_{k-2} & 0 \end{pmatrix}$ for some $k \geq 2$. Hence we have

$$\begin{aligned} (F_3)^{k+1} &= F_3(F_3)^k = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_{k+1} & f_k & 0 \\ f_k & f_{k-1} & 0 \\ f_{k-1} & f_{k-2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} f_{k+1} + f_k & f_k + f_{k-1} & 0 \\ f_{k+1} & f_k & 0 \\ f_k & f_{k-1} & 0 \end{pmatrix} = \begin{pmatrix} f_{k+2} & f_{k+1} & 0 \\ f_{k+1} & f_k & 0 \\ f_k & f_{k-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} f_{(k+1)+1} & f_{k+1} & 0 \\ f_{k+1} & f_{(k+1)-1} & 0 \\ f_{(k+1)-1} & f_{(k+1)-2} & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$(F_3)^n = \begin{pmatrix} f_{n+1} & f_n & 0 \\ f_n & f_{n-1} & 0 \\ f_{n-1} & f_{n-2} & 0 \end{pmatrix}$$

for integer $n \geq 2$. $\square$

Let

$$F_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 & 0 & 0 \\ f_1 & f_0 & 0 & 0 \\ f_0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4}$$

Using the same argument as in the case of 3 by 3 , we have the following result.

Proposition 4. *Let $\{f_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers. If F_4 is defined as in (4), then* $(F_4)^n = \begin{pmatrix} f_{n+1} & f_n & 0 & 0 \\ f_n & f_{n-1} & 0 & 0 \\ f_{n-1} & f_{n-2} & 0 & 0 \\ f_{n-2} & f_{n-3} & 0 & 0 \end{pmatrix}$ *for $n \geq 3$.*

We now consider, for some $m \geq 5$, the m by m matrix

$$F_m = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 & 0 & 0 & \cdots & 0 & 0 \\ f_1 & f_0 & 0 & 0 & \cdots & 0 & 0 \\ f_0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{5}$$

Then the recurrence relation $U_{n+1} := F_m U_n$ will result in $u_{n+1,1} = u_{n,1} + u_{n,2}$ and for $j = 2, 3, \ldots, m$ we have $u_{n+1,j} = u_{n,j-1}$. Let $F_m(:, k)$ be the k^{th} column of F_m, then

$$(F_m)^{m-2}F_m(:,1) = \begin{pmatrix} f_m \\ f_{m-1} \\ f_{m_2} \\ \vdots \\ f_1 \end{pmatrix}, (F_m)^{m-2}F_m(:,2) = \begin{pmatrix} f_{m-1} \\ f_{m-2} \\ f_{m_3} \\ \vdots \\ f_0 \end{pmatrix} \text{ and } (F_m)^{m-2}F(:,k) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for $k = 3, \ldots, m$. This implies

$$(F_m)^{m-1} = \begin{pmatrix} f_m & f_{m-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-1} & f_{m-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-2} & f_{m-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-3} & f_{m-4} & 0 & 0 & \cdots & 0 & 0 \\ f_{m-4} & f_{m-5} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1 & f_0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{6}$$

We now have the following result.

Theorem 5. *Let $\{f_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers. If F_m, for some $m \geq 3$, is the m by m matrices defined in (5), then*

$$(F_m)^n = \begin{pmatrix} f_{n+1} & f_n & 0 & 0 & \cdots & 0 & 0 \\ f_n & f_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-1} & f_{n-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-2} & f_{n-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-3} & f_{m-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-(m-2)} & f_{n-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

for $n \geq m-1$.

Proof. We prove by induction on n. Equation (6) guarantees that theorem is true for $n = m-1$. Assume now that $(F_m)^k = \begin{pmatrix} f_{k+1} & f_k & 0 & 0 & \cdots & 0 & 0 \\ f_k & f_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-1} & f_{k-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-2} & f_{k-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-3} & f_{k-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{k-(m-2)} & f_{k-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ for $k \geq m-1$. Then

$$(F_m)^{k+1} = F_m(F_m)^k = \begin{pmatrix} f_{k+1}+f_k & f_k+f_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k+1} & f_k & 0 & 0 & \cdots & 0 & 0 \\ f_k & f_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-1} & f_{k-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{k-2} & f_{k-3} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{k-(m-3)} & f_{k-(m-2)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and hence

$$(F_m)^{k+1} = \begin{pmatrix} f_{(k+1)+1} & f_{k+1} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)} & f_{(k+1)-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)-1} & f_{(k+1)-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)-2} & f_{(k+1)-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{(k+1)-3} & f_{(k+1)-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{(k+1)-(m-2)} & f_{(k+1)-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Therefore,

$$(F_m)^n = \begin{pmatrix} f_{n+1} & f_n & 0 & 0 & \cdots & 0 & 0 \\ f_n & f_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-1} & f_{n-2} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-2} & f_{n-3} & 0 & 0 & \cdots & 0 & 0 \\ f_{n-3} & f_{m-4} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{n-(m-2)} & f_{n-(m-1)} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

for $n \geq m-1$. □

We note that first two columns the matrix $(F_m)^n$ for some $n \geq m-1$ store $m+1$ distinct consecutive terms of the Fibonacci numbers.

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References

[1] Dunlap R A 1997 *The Golden Ratio and Fibonacci Numbers*(World Scientific Publishing : Singapore).
[2] Hawkins K, Johnson U and Mathes B 2015 *Linear Algebra and its Applications*, **475** 80–89.
[3] Luis R and Oliveira H M 2014 *Applied Mathematics and Computation*, **226**, 101–116.
[4] Meinke A M 2011 *Fibonacci Numbers And Associated Matrices* (Thesis Kent State University).