

# Bounds for scrambling index of primitive graphs

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**Abstract.** A connected graph is primitive provided there is a positive integer  $m$  such that for each pair of vertices  $u$  and  $v$  there is a walk of length  $m$  connecting  $u$  and  $v$ . The scrambling index of a primitive graph  $G$  is the smallest positive integer  $k$  such that for each pair of vertices  $u$  and  $v$  there is a vertex  $w$  such that there exist a walk of length  $k$  connecting  $u$  and  $w$  and a walk of length  $k$  connecting  $v$  and  $w$ . For a primitive graph  $G$  with smallest cycle  $C_s$  of length  $s$ , we present an upper bound on the scrambling index of  $G$  that depends on  $s$  and the maximum distance between vertices in  $G$  and the cycle  $C_s$ . We then classify the graphs that satisfy the upper bound.

## 1. Introduction

Let  $G(V, E)$  denote a simple graph on  $n$  vertices. We follow [1, 2] for terminologies on graph. A walk  $W$  connecting  $u$  and  $v$  is denoted by  $W_{uv}$ . A walk connecting  $u$  and  $v$  is closed whenever  $u = v$ , and is open otherwise. A path  $P_{uv}$  connecting  $u$  and  $v$  is a walk  $W_{uv}$  with distinct vertices, except possibly  $u = v$ . A cycle is a closed path. The length of a walk  $W_{uv}$  by  $\ell(W_{uv})$ .

A walk  $W_{uv}$  is also denoted by  $u \overset{W_{uv}}{--} v$ . For simplicity a walk of length  $k$  connecting  $u$  and  $v$  is denoted by  $u \overset{k}{--} v$  walk. The distance between vertex  $u$  and vertex  $v$  in a connected graph, denoted  $d(u, v)$ , is the length of the shortest path connecting  $u$  and  $v$ . For any set  $X \subseteq V(G)$  and a vertex  $v \notin X$  the distance between  $v$  and  $X$  is defined by

$$d(v, X) = \min\{d(v, x) : x \in X\}.$$

If  $v \in X$  we define  $d(v, X) = 0$ .

A connected graph  $G$  is primitive if there is a positive integer  $k$  such that for each pair of vertices  $u$  and  $v$  in  $G$ , there is a  $u \overset{k}{--} v$  walk. The least of such positive integer  $k$  is the exponent of  $G$  and is denoted by  $\exp(G)$ . It is well known (see e.g. [1]) that a graph  $G$  is primitive if and only if  $G$  has a cycle of odd length.

In 2009, Akelbek and Kirkland [3, 4] introduced the notion of scrambling index of graph. The *scrambling index* of a primitive graph  $G$ , denoted by  $k(G)$ , is the least positive integer  $k$  such that for any pair of distinct vertices  $u$  and  $v$  in  $G$  there exists a vertex  $w$  with the property that there is a  $u \overset{k}{--} w$  walk and a  $v \overset{k}{--} w$  walk. For a pair of distinct vertices  $u$  and  $v$  in  $G$  the local scrambling index of  $u$  and  $v$  is the number

$$k_{u,v}(G) = \min_{w \in V(G)} \{k : \text{there are } u \overset{k}{--} w \text{ walk and } v \overset{k}{--} w \text{ walk}\}.$$



We note that if the local scrambling index of  $u$  and  $v$  is  $k_{u,v}(G)$ , then for any positive integer  $\ell \geq k_{u,v}(G)$  we can find a vertex  $w'$  such that there is a walk  $u \overset{\ell}{-} w'$  and  $v \overset{\ell}{-} w'$ . This implies

$$k(G) = \max_{u,v \in V(G)} \{k_{u,v}(G)\}.$$

Chen and Liu [5] have shown that for a primitive graph  $G$  on  $n$  vertices with the smallest cycle of length  $s$ ,  $k(G) \leq (s-1)/2 + n - s$ . The purpose of this paper is to explore a different upper bound from Chen and Liu's bound, which in most case will be smaller than  $(s-1)/2 + n - s$ . In Section 2 we discuss some important properties of  $u-v$  walks. In Section 3 we present an upper bound for the scrambling index of primitive graphs. Finally, in Sections 4 we discuss classes of primitive graphs whose scrambling index achieved the upper bound.

## 2. Properties of walks

In this section we discuss some properties of  $u-v$  walk with end vertices  $u$  and  $v$  in some primitive graph.

**Theorem 1.** [2] *Let  $G$  be a graph. Every  $u-v$  walk in  $G$  contains a  $u-v$  path.*

Theorem 1 basically saying that we can shorten a  $u-v$  walk into a shorter  $u-v$  walk. The following result guarantees that we can lengthen a  $u-v$  walk into a longer  $u-v$  walk with the same parity.

**Proposition 2.** *Let  $G$  be a graph and  $u$  and  $v$  be two vertices in  $G$ . Every  $u \overset{t}{-} v$  walk can be extended to a  $u \overset{t+2m}{-} v$  walk for some positive integer  $m$ .*

*Proof.* Let  $u$  and  $v$  be vertices in  $G$  and let

$$W_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_{t-1} - v_t = v$$

be a  $u \overset{t}{-} v$  walk in  $G$  and let  $W_2 : v - v_{t-1} - v$  be a closed walk of length 2. Then the walk

$$W'_{uv} : u \overset{W_{uv}}{-} v \overset{mW_2}{-} v$$

that starts at  $u$ , moves to  $v$  along the walk  $W_{uv}$ , and then moves  $m$  times around the closed walk  $W_2 : v - v_{t-1} - v$  is a  $u \overset{t+2m}{-} v$  walk.  $\square$

**Proposition 3.** *Let  $G$  be a graph. Then there is a  $u \overset{2t}{-} v$  walk in  $G$  if and only if there is vertex  $w$  in  $G$  such that there are a  $u \overset{t}{-} w$  walk and a  $v \overset{t}{-} w$  walk in  $G$ .*

*Proof.* Suppose there is a vertex  $w$  in  $G$  such that there are a  $u \overset{t}{-} w$  walk and a  $v \overset{t}{-} w$  walk in  $G$ . Then there is a  $w \overset{t}{-} v$  walk in  $G$ . This implies the walk  $W_{uv} : u \overset{t}{-} w \overset{t}{-} v$  is a  $u \overset{2t}{-} v$  walk.

Assume now that

$$W_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_{2t-1} - v_{2t} = v$$

is a  $u \overset{2t}{-} v$  walk in  $G$ . If we choose  $w = v_t$ , then there is a vertex  $w$  such that there are a  $u \overset{t}{-} w$  walk and a  $v \overset{t}{-} w$  walk in  $G$ .  $\square$

### 3. Bounds for scrambling index

We present a lower bound and an upper for scrambling index of primitive graphs.

**Theorem 4.** *Let  $G$  be a primitive graph and let  $C_s$  be a cycle of odd length  $s$ . If  $G$  does not have odd cycles with length smaller than  $s$ , Then*

$$k(G) \leq \frac{s-1}{2} + \max_{v \in V(G)} \{d(v, V(C_s))\}.$$

*Proof.* For each pair of distinct vertices  $u$  and  $v$  we show that there is a  $u-v$  walk of length  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ .

First, we claim that for any distinct vertices  $u$  and  $v$  in  $G$ , there is a path  $P_{uv}$  such that  $\ell(P_{uv}) \leq (s-1)/2 + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . If both  $u$  and  $v$  lie on the cycle  $C_s$ , then there is a path  $P_{uv}$  with  $\ell(P_{uv}) \leq (s-1)/2$ . If  $v$  lies in  $V(G) \setminus V(C_s)$  and  $u$  lies on  $C_s$ , then there is a path  $P_{uv}$  with  $\ell(P_{uv}) \leq \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Suppose  $u$  and  $v$  lie on  $V(G) \setminus V(C_s)$ . Assume without loss of generality that  $d(u, V(C_s)) \geq d(v, V(C_s))$  and that  $d(u, V(C_s))$  is obtained by a path  $P_{uc}$  for some vertex  $c$  in  $C_s$ . If the vertex  $v$  lies on the path  $P_{uc}$ , then  $\ell(P_{uv}) \leq \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Otherwise, there is a vertex  $y$  in  $C_s$  such that  $d(v, y) = d(v, V(C_s))$ . Since  $d(u, c), d(v, y) \leq \max_{v \in V(G)} \{d(v, V(C_s))\}$ , then the walk

$$W_{uv} : u \xrightarrow{P_{u,c}} c \xrightarrow{P_{c,y}} y \xrightarrow{P_{y,v}} v$$

is a  $u-v$  walk of length  $\ell(W_{uv}) \leq (s-1)/2 + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . By Theorem 1 there is a path  $P_{uv}$  of length  $\ell(P_{uv}) \leq (s-1)/2 + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ .

We now show that the path  $P_{uv}$  can be extended to a  $u-v$  walk of length exactly  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . If there is a  $u-v$  path  $P_{uv}$  of even length, then Proposition 2 guarantees that we can extend the path  $P_{uv}$  to a  $u-v$  walk of length exactly  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . We now assume that all  $u-v$  paths are of odd lengths. Notice that  $u$  and  $v$  cannot be both on  $C_s$ . We show that we can extend a  $u-v$  path into a  $u-v$  walk of length exactly  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . We consider two cases.

**Case 1:** *There exists a path  $P_{uv}$  that has vertices in common with the cycle  $C_s$*

We claim that the path  $P_{uv}$  and the cycle  $C_s$  have exactly one vertex in common. Suppose on the contrary that  $P_{uv}$  and  $C_s$  have more than one vertex in common. Let  $x_0$  and  $y_0$  be two vertices on  $C_s$  that lie on the path  $P_{uv}$ . We note that there are two paths on  $C_s$  say  $P_{x_0, y_0}$  and  $P'_{x_0, y_0}$  connecting  $x_0$  and  $y_0$ . Since  $C_s$  is of odd length,  $\ell(P_{x_0, y_0}) \not\equiv \ell(P'_{x_0, y_0}) \pmod{2}$ . This implies

either the path  $P_{uv} : u \xrightarrow{P_{x_0, y_0}} x_0 \xrightarrow{P_{x_0, y_0}} y_0 \xrightarrow{P'_{x_0, y_0}} v$  or the path  $P_{uv} : u \xrightarrow{P_{x_0, y_0}} x_0 \xrightarrow{P'_{x_0, y_0}} y_0 \xrightarrow{P_{x_0, y_0}} v$  is a  $u-v$  path of even length. This contradicts the fact that all  $u-v$  paths in  $G$  are of odd lengths.

We show that there is a  $u-v$  walk  $W_{uv}$  of even length  $\ell(W_{uv}) \leq (s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Suppose the path  $P_{uv}$  and the cycle  $C_s$  have a vertex in common at

$v_0$ . If  $v_0 = u$ , then the walk  $W_{uv} : u = v_0 \xrightarrow{C_s} u = v_0 \xrightarrow{P_{u,v}} v$  is a  $u-v$  walk of even length

$\ell(W_{uv}) \leq (s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Similarly, if  $v_0 = v$ , then the walk  $W_{uv} : u \xrightarrow{P_{u,v_0}} v_0 \xrightarrow{C_s} v = v_0$  is a  $u-v$  walk of even length  $\ell(W_{uv}) \leq (s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ .

If  $u \neq v_0$  and  $v \neq v_0$ , then the  $u-v$  path  $P_{uv}$  can be decomposed into  $u-v_0$  path  $P_{u,v_0}$  and  $v_0-v$  path  $P_{v_0,v}$ . Since all paths  $P_{uv}$  are of odd length,  $\ell(P_{u,v_0}) + \ell(P_{v_0,v}) \leq$

$2 \max_{v \in V(G)} \{d(v, V(C_s))\} - 1$ . This implies the walk  $W_{uv} : u \xrightarrow{P_{u,v_0}} v_0 \xrightarrow{C_s} v_0 \xrightarrow{P_{v_0,v}} v$  is a  $u-v$  walk of even length  $\ell(W_{uv}) \leq (s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Proposition 2 guarantees that we can extend the  $u-v$  walk  $W_{uv}$  into a  $u-v$  walk with length exactly

$$(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}.$$

**Case 2:** All paths  $P_{uv}$  and the cycle  $C_s$  have no vertices in common

There is a path  $P_{uv}$  and a vertex  $x_0$  on  $P_{uv}$  and a vertex  $y_0$  on  $C_s$  such that

$$d(x_0, y_0) = \min\{d(x, y) : x \text{ on } P_{uv} \text{ and } y \text{ on } C_s\}.$$

Notice that the path  $P_{uv}$  can be decomposed into a path  $P_{u,x_0}$  connecting  $u$  and  $x_0$ , and a path  $p_{x_0,v}$  connecting  $x_0$  and  $v$ . Since  $\ell(P_{uv})$  is odd, the walk

$$u \xrightarrow{P_{u,x_0}} x_0 \xrightarrow{P_{x_0,y_0}} y_0 \xrightarrow{C_s} y_0 \xrightarrow{P_{y_0,x_0}} x_0 \xrightarrow{P_{x_0,v}} v$$

is a  $u-v$  walk of even length. Notice also that  $\ell(P_{u,x_0}) \not\equiv \ell(P_{x_0,v}) \pmod{2}$ . This implies  $\ell(P_{u,y_0}) \not\equiv \ell(P_{y_0,v}) \pmod{2}$ . Since  $\ell(P_{u,y_0}), \ell(P_{y_0,v}) \leq \max_{v \in V(G)} \{d(v, V(C_s))\}$ , we have  $\ell(P_{u,y_0}) + \ell(P_{y_0,v}) \leq 2 \max_{v \in V(G)} \{d(v, V(C_s))\} - 1$ . Thus  $\ell(W_{uv}) \leq s-1 + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Proposition 2 guarantees that there is a  $u-v$  walk  $W_{uv}$  with length equals  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$

Now Proposition 3 guarantees that for each pair of vertices  $u$  and  $v$  there is a vertex  $w$  such that there is a  $u \xrightarrow{t} w$  walk and there is a  $v \xrightarrow{t} w$  walk with  $t = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Thus the scrambling index  $k(G) \leq (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ .  $\square$

We note that for a primitive graph  $G$  with the smallest odd cycle of length  $s$ ,  $\max_{v \in V(G)} \{d(v, V(C_s))\} \leq n-s$ . Hence the bound given in Theorem 4 is smaller than or equal to the bound  $(s-1)/2 + (n-s)$ .

#### 4. Primitive graphs achieving the upper bound

In this section we discuss classes of primitive graphs that satisfy the upper bound given in Theorem 4. We first characterize and then discuss instances of such primitive graphs.

**Corollary 5.** Let  $G$  be a primitive graph and let  $C_s$  be a smallest odd cycle of length  $s$  in  $G$ . The scrambling index

$$k(G) = \frac{s-1}{2} + \max_{v \in V(G)} \{d(v, V(C_s))\}$$

if and only if there are vertices  $u_0$  and  $v_0$  such that the shortest even  $u_0-v_0$  walk is a walk of length  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ .

*Proof.* Assume that there are two distinct vertices  $u_0$  and  $v_0$  in  $G$  such that the shortest even walk connecting  $u_0$  and  $v_0$  is of length  $s-1 + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Then by

Proposition 3 there exists a vertex  $w$  such that there is a  $u_0 \xrightarrow{t} w$  walk and a  $v_0 \xrightarrow{t} w$  walk with  $t = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ . We claim that  $k_{u_0,v_0}(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Suppose on the contrary that  $k_{u_0,v_0}(G) = \ell$  for some positive integer  $\ell < (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Then there exists a vertex  $w'$  with the property that there is a  $u_0 \xrightarrow{\ell} w'$  walk and a  $v_0 \xrightarrow{\ell} w'$  walk. But this implies the  $u_0-v_0$  walk  $W_{u_0,v_0} : u_0 \xrightarrow{\ell} w' \xrightarrow{\ell} v_0$  is a  $u_0-v_0$  walk of even length  $2\ell < (s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . This contradicts the fact that the shortest even  $u_0-v_0$  walk is of length  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Therefore,

$$k_{u_0,v_0}(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$$

and hence

$$k(G) \geq k_{u_0, v_0}(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}.$$

By Theorem 4 we conclude that  $k(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ .

We now assume that  $k(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Then there are two distinct vertices  $u_0$  and  $v_0$  such that

$$k_{u_0, v_0}(G) = \min_{w \in V(G)} \{t : \text{there are } u_0 \overset{t}{\dashrightarrow} w \text{ and } v_0 \overset{t}{\dashrightarrow} w \text{ walks}\} = k(G).$$

This implies the walk  $u_0 \overset{k(G)}{\dashrightarrow} w \overset{k(G)}{\dashrightarrow} v_0$  is the shortest walk of even length  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$  connecting  $u_0$  and  $v_0$ .  $\square$

We next discuss the class of primitive graph with smallest cycle of length  $s$  that satisfies the bound in Theorem 4. For that purpose we need the following definition. An open path  $P$  of length  $\ell(P) = \max_{v \in V(G)} \{d(v, V(C_s))\}$  with one end vertex in  $C_s$  is called to be *special* if it does not have vertices in common with cycle of odd length other than  $C_s$ .

**Corollary 6.** *Let  $G$  be a primitive graph with the shortest odd cycle of length  $s$ . If  $G$  has a special path, then  $k(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ .*

*Proof.* Let  $P_{v_0, u_0}$  be a special path in  $G$  and let  $u_0 \in V(C_s)$  and  $v_0 \in V(G)$ . Let  $v_0 - y_0$  be an edge of  $P_{v_0, u_0}$  such that  $d(v_0, u_0) > d(y_0, u_0)$ . Then the  $v_0 - y_0$  walk  $W_{v_0, y_0} : v_0 \overset{P_{v_0, u_0}}{\dashrightarrow} u_0 \overset{C_s}{\dashrightarrow} u_0 \overset{P_{u_0, y_0}}{\dashrightarrow} y_0$  is the shortest walk connecting  $v_0$  and  $y_0$  of even length  $(s-1) + 2 \max_{v \in V(G)} \{d(v, V(C_s))\}$ . Corollary 5 implies that  $k(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ .  $\square$

**Corollary 7.** *Let  $G$  be a primitive graph containing a unique cycle of odd length. Let  $C_s$  be the odd cycle in  $G$  say of length  $s$ . Then  $k(G) = (s-1)/2 + \max_{v \in V(G)} \{d(v, V(C_s))\}$ .*

*Proof.* Since  $G$  has only one cycle of odd length,  $G$  must contain a special path. The conclusion follows from Corollary 6.  $\square$

Let  $G$  be a loopless primitive graphs on  $n$  vertices with the smallest cycle of length  $s$ . Since  $s \geq 3$  and  $\max_{v \in V(G)} \{d(v, V(C_s))\} \leq n-3$ , then by Theorem 4 we have  $k(G) \leq n-2$ . Let  $SI_n$  denote the set of positive integers  $t$  for which there exists a loopless primitive graph on  $n$  vertices with scrambling index equals  $t$ .

**Corollary 8.** *For any positive integer  $n \geq 3$ ,  $SI_n = \{1, 2, \dots, n-2\}$ .*

*Proof.* For positive integer  $t$ ,  $3 \leq t \leq n-1$ , we define a primitive graph  $G_t$  on  $n$  vertices  $\{v_1, v_2, \dots, v_n\}$  to be the graph with edge set

$$\begin{aligned} E(G_t) = & \{v_1 - v_2 - v_3 - v_1\} \cup \{v_3 - v_4 - \dots - v_{t-1} - v_t\} \\ & \cup \{v_t - v_{t+i} : i = 1, 2, \dots, n-t\} \end{aligned}$$

as shown in Figure 1.

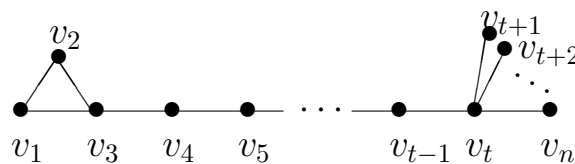


Figure 1. The graph  $G_t$

The graph  $G_t$  has a unique odd cycle of length 3. Moreover,  $G_t$  has special paths of length  $t-2$ . Corollary 6 implies that  $k(G_t) = t-1$ . Since  $3 \leq t \leq n-1$  and  $k(K_n) = 1$ , then for each positive integer  $1 \leq p \leq n-2$ , there exists primitive graph  $k(G) = p$ . Hence  $SI_n = \{1, 2, \dots, n-2\}$ .  $\square$

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