

Scrambling index of certain primitive graphs consisting of two disjoint odd cycles connected by some paths

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Abstract. The scrambling index of a primitive graph G is the smallest positive integer k such that for each pair of vertices u and v there is a vertex w such that there exists a uw -walk and a vw -walk of length k . We discuss the scrambling index of primitive graphs G consisting of two disjoint odd cycles each of length s connected by some paths of length ℓ_0 . For such primitive graphs G we present formulae for scrambling indices that depend on s and ℓ_0 .

1. Introduction

Let $G(V, E)$ be a graph. Let u and v be any vertices in G , a uv -walk W_{uv} of length t connecting u and v is a sequence of vertices $u = u_0, u_1, u_2, \dots, u_t = v$ and a sequence of edges

$$\{u_0, u_1\}, \{u_1, u_2\}, \dots, \{u_{t-1}, u_t\},$$

where the vertices and the edges are not necessarily distinct. A walk W connecting u and v is denoted by W_{uv} . A walk connecting u and v is closed whenever $u = v$, and is open otherwise. A path P_{uv} connecting u and v is a walk W_{uv} with distinct vertices, except possibly $u = v$. A cycle is a closed path. The length of a walk W_{uv} is the number of edges in W_{uv} , and is denoted by $\ell(W_{uv})$. A walk W_{uv} with sequence of edges $\{u_0, u_1\}, \{u_1, u_2\}, \dots, \{u_{t-1}, u_t\}$ is also denoted by

$$u = u_0 - u_1 - u_2 - \dots - u_{t-1} - u_t = v.$$

By an odd (even) cycle we mean a cycle of odd (even) length.

A graph G is connected provided that for each pair of vertices u and v in G there is uv -walk in G . The distance between vertices u and v in a connected graph G , denoted $d(u, v)$, is the length of a shortest path connecting u and v . A connected graph G is primitive provided there is a positive integer m such that for each pair of vertices u and v in G , there is a uv -walk of length m . The smallest of such positive integer m is the exponent of G and is denoted by $\exp(G)$. It is well known (see e.g [1]) that a graph G is primitive if only if G has a cycle of odd length.

In 2009, Akelbek and Kirkland [2, 3] introduced the notion of scrambling index of graph. The *scrambling index* of a primitive graph G , denoted by $k(G)$, is the smallest positive integer k such that for any pair of distinct vertices u and v in G there is a vertex w such that there is a uw -walk



and a vw -walk of length k . Chen and Liu [4] have shown that for a primitive graph G on n vertices with the smallest cycle of length s , then $k(G) \leq n - (s + 1)/2$. They also characterize primitive graph that attains the upper bound $n - (s + 1)/2$.

This paper discuss the scrambling index of a class of primitive graph consisting of two disjoint odd cycles connected by some paths. For this purpose, we first introduce some preliminaries in setting up a lower and upper bound for scrambling index. Then we discuss the scrambling index of barbell and finally we discuss scrambling index of a prism.

2. Background

For a pair of distinct vertices u and v in G the local scrambling index of u and v is the number

$$k_{u,v}(G) = \min_{w \in V(G)} \{k : \text{there are } uw\text{-walk and } vw\text{-walk of length } k\}.$$

We note that if the local scrambling index of u and v is $k_{u,v}(G)$, then for any positive integer $\ell \geq k_{u,v}(G)$ we can find a vertex w' such that there is a walk uw' -walk and a vw' -walk of length ℓ . This implies

$$k(G) = \max_{u,v \in V(G)} \{k_{u,v}(G)\}.$$

Proposition 1. *Let G be a primitive graph and k' be an even positive integer. If for each pair of vertices u and v in G there is an even walk W_{uv} of length $\ell(W_{uv}) \leq k'$, then $k(G) \leq k'/2$.*

Proof. Let u and v be any two vertices in G and let W_{uv} be an even walk $u = v_0 - v_1 - v_2 - \dots - v_{2m-1} - v_{2m} = v$ for some positive integer m of length $\ell(W_{uv}) \leq k'$. Let C_2 be the closed walk $v_{2m} - v_{2m-1} - v_{2m}$ of length 2. Then the walk W'_{uv} that starts at u , moves to v along the walk W_{uv} and then moves $k' - \ell(W_{uv})$ times around the cycle C_2 is a uv -walk of length k' . Since k' is even, there is a vertex w such that there is a uw -walk of length $k'/2$ and there is a vw -walk of length $k'/2$. Hence $k(G) \leq k'/2$. \square

3. Preliminary results

Let C_1 and C_2 be disjoint odd cycles each of length s . Let P be a path of length ℓ_0 with one end vertex on C_1 and the other end on C_2 . A connected graph G consisting of two disjoint odd cycles C_1 and C_2 connected by a path P of length ℓ_0 is called an (s, ℓ_0) -barbell. Let x_1 be a vertex in common to C_1 and P and let x_2 be a vertex in common C_2 and P . We note that $d(x_1, x_2) = \ell_0$. For each vertex $u \in C_1$, let P_{ux_1} be the shortest ux_1 -path and let P'_{ux_1} be the ux_1 -path of length $s - d(u, x_1)$. Similarly, for each vertex $u \in C_2$, let P_{ux_2} be the shortest ux_2 -path and let P'_{ux_2} be the ux_2 -path of length $s - d(u, x_2)$.

Lemma 2. *Let $s \geq 5$ be an odd integer and G be an (s, ℓ_0) -barbell. If ℓ_0 is odd, then $k(G) = (s + \ell_0)/2$.*

Proof. Let u and v be the two median vertices on P such that $d(u, x_1) = (\ell_0 - 1)/2$ and $d(v, x_2) = (\ell_0 - 1)/2$. Notice that $d(u, v) = 1$. Then the shortest even uv -walk is the walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path, then moves one time around the cycle C_1 and back at x_1 , finally moves to v along the x_1v -path. We note that $\ell(W_{uv}) = s + \ell_0$. Hence $k(G) \geq (s + \ell_0)/2$.

By Proposition 1 it remains to show that for each pair of vertices u and v there is an even walk W_{uv} of length $\ell(W_{uv}) \leq s + \ell_0$. If $d(u, v)$ is even then we are done. So we assume that $d(u, v)$ is odd.

Case 1. Both vertices $u, v \in C_1$ or $u, v \in C_2$. Since u and v both lie on the same cycle, then there is an even uv -path of length $s - d(u, v) < s + \ell_0$.

Case 2. The vertex $u \in C_1$ and $v \in C_2$. We note that $d(u, v) = d(u, x_1) + \ell_0 + d(x_2, v)$. Assume without loss of generality that $d(u, x_1) \leq d(v, x_2)$. Then the uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $d(u, x_1)$, then moves to x_2 along the path P , and finally moves to v along the vx_2 -path of length $s - d(v, x_2)$ is an even uv -walk of length $s + \ell_0 + d(u, x_1) - d(v, x_2)$. Since $d(u, x_1) \leq d(v, x_2)$, we have $\ell(W_{uv}) \leq s + \ell_0$.

Case 3. The vertex $u \in C_1$ and $v \in P$ or $u \in C_2$ and $v \in P$. We may assume that $u \in C_1$. Notice that $d(u, v) = d(u, x_1) + d(x_1, v)$ is odd. Then the uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $s - d(u, x_1)$, and finally moves to v along the x_1v -path is an even uv -walk of length $\ell_0(W_{uv}) = s - d(u, x_1) + d(x_1, v) \leq s + \ell_0$.

Case 4. The vertices $u, v \in P$. We assume that $d(x_1, u) < d(x_1, v)$. If $d(u, x_1) \leq d(v, x_2)$, then $d(u, x_1) \leq (\ell_0 - 1)/2$. The uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $d(u, x_1)$, then moves one time around the cycle C_1 and back at x_1 , then moves to u along the x_1u -path of length $d(u, x_1)$, and finally moves to v along the uv -path of length $d(u, v)$, is an even uv -walk. Notice that $\ell_0(W_{uv}) = s + 2d(u, x_1) + d(u, v)$. Since $\ell_0 = d(x_1, u) + d(u, v) + d(v, x_2)$ and $d(u, x_1) \leq d(v, x_2)$, we have $\ell(W_{uv}) \leq s + \ell_0$.

If $d(u, x_1) > d(v, x_2)$, then $d(v, x_2) < (\ell_0 - 1)/2$. The uv -walk W_{uv} that starts at u , moves to v along the uv -path of length $d(u, v)$, then moves to x_2 along the vx_2 -path of length $d(v, x_2)$, then moves one time around the cycle C_2 and back at x_2 , and finally moves to v along the x_2v -path of length $d(v, x_2)$, is an even uv -walk. Notice that $\ell(W_{uv}) = d(u, v) + 2d(v, x_2) + s$. Since $\ell_0 = d(x_1, u) + d(u, v) + d(v, x_2)$ and $d(u, x_1) > d(v, x_2)$, we have $\ell(W_{uv}) < s + \ell_0$.

Hence for any pair of distinct vertices u and v there is a uv -walk W_{uv} of length $\ell_0(W_{uv}) \leq s + \ell_0$. Proposition 1 guarantees that $k(G) \leq (s + \ell_0)/2$. \square

Lemma 3. Let $s \geq 5$ be an odd integer and G be an (s, ℓ_0) -barbell. If ℓ_0 is even, then $k(G) = (s + \ell_0 - 1)/2$.

Proof. Let v be the median vertex on P and let u be the vertex such that $d(u, x_1) = d(v, x_1) - 1$. Then $d(v, x_1) = \ell_0/2$ and $d(u, v) = 1$. Notice that the shortest even uv -walk is the walk that starts at u , moves to x_1 along the path of length $\ell_0/2 - 1$, then moves one time around the cycle C_1 and back at x_1 , then moves to u along the path of length $\ell_0/2 - 1$, and finally moves to v along the path of length ℓ_0 . Therefore $\ell(W_{uv}) = s + \ell_0 - 1$ and hence $k(G) \geq (s + \ell_0 - 1)/2$.

By Proposition 1 it remains to show that for each pair of vertices u and v there is an even walk W_{uv} of length $\ell(W_{uv}) \leq s + \ell_0 - 1$. If $d(u, v)$ is even then we are done. So we assume that $d(u, v)$ is odd.

Case 1. Both vertices $u, v \in C_1$ or $u, v \in C_2$. As in the proof of Lemma 2 there is a uv -walk of lengths $d(u, v) \leq s + \ell_0 - 1$.

Case 2. The vertex $u \in C_1$ and $v \in C_2$. We note that $d(u, v) = d(u, x_1) + \ell_0 + d(x_2, v)$. Since ℓ_0 is odd, we assume without loss of generality that $d(u, x_1) < d(v, x_2)$. Then the uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $d(u, x_1)$, then moves to x_2 along the path P , and finally moves to v along the vx_2 -path of length $s - d(v, x_2)$ is an even uv -walk of length $s + \ell_0 + d(u, x_1) - d(v, x_2)$. Since $d(u, x_1) < d(v, x_2)$, we have $\ell(W_{uv}) \leq s + \ell_0 - 1$.

Case 3. The vertex $u \in C_1$ and $v \in P$ or $u \in C_2$ and $v \in P$. We may assume that $u \in C_1$. Notice that $d(u, v) = d(u, x_1) + d(x_1, v)$ is odd. Then the uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $s - d(u, x_1)$, and finally moves to v along the x_1v -path is an even uv -walk of length $\ell(W_{uv}) = s - d(u, x_1) + d(x_1, v) \leq s + \ell_0$. Since ℓ_0 is even, then $\ell(W_{uv}) \leq s + \ell_0 - 1$.

Case 4. The vertices $u, v \in P$. We assume that $d(x_1, u) < d(x_1, v)$. If $d(u, x_1) \leq d(v, x_2)$, then $d(u, x_1) \leq \ell_0/2 - 1$. The uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $d(u, x_1)$, then moves one time around the cycle C_1 and back at x_1 , then moves to u along the x_1u -path of length $d(u, x_1)$, and finally moves to v along the uv -path of length $d(u, v)$ is an

even uv -walk. Notice that $\ell(W_{uv}) = s + 2d(u, x_1) + d(u, v)$. Since $\ell_0 = d(u, x_1) + d(u, v) + d(v, x_2)$, and $d(u, x_1) < d(v, x_2)$, we have $\ell(W_{uv}) \leq s + \ell_0 - 1$.

If $d(u, x_1) > d(v, x_2)$, then the uv -walk W_{uv} that starts at u , moves to v along the uv -path of length $d(u, v)$, then moves to x_2 along the vx_2 -path of length $d(v, x_2)$, then moves one time around the cycle C_2 and back at x_2 , and finally moves to v along the x_2v -path of length $d(v, x_2)$, is an even uv -walk. Notice that $\ell(W_{uv}) = d(u, v) + 2d(v, x_2) + s$. Since $\ell_0 = d(u, x_1) + d(u, v) + d(v, x_2)$ and $d(u, x_1) > d(v, x_2)$, we have $\ell(W_{uv}) \leq s + \ell_0 - 1$.

Hence for any pair of distinct vertices u and v there is a uv -walk W_{uv} of length $\ell_0(W_{uv}) \leq s + \ell_0 - 1$. Proposition 1 guarantees that $k(G) \leq (s + \ell_0 - 1)/2$. \square

Combining Lemma 2 and Lemma 3 we have the following result.

Theorem 4. *Let $s \geq 5$ be an odd integer and G be an (s, ℓ_0) -barbell. Then $k(G) = \lceil (s + \ell_0 - 1)/2 \rceil$.*

4. Main results

Let C_1 and C_2 be cycles each of odd length s . Let $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$ such that $d(x_1, y_1) = d(x_2, y_2)$ and let P_1 be an x_1x_2 -path and P_2 be a y_1y_2 -path each of length ℓ_0 . An (s, ℓ_0) -two bar barbell is a graph consisting of disjoint cycles C_1 and C_2 connected by two paths P_1 and P_2 .

Lemma 5. *Let $s \geq 5$ be an odd integer and G be an (s, ℓ_0) -two bar barbell. If ℓ_0 is odd, then $k(G) = (s + \ell_0)/2$.*

Proof. Let u' and v' be the two median vertices on P_1 . Then the shortest even $u'v'$ -walk is of length $s + \ell_0$. Hence $k(G) \geq (s + \ell_0)/2$. Let u and v be two vertices in G . We may assume that $d(u, v)$ is odd. Considering the proof of Lemma 2 it remains to show that if $u \in P_1$ and $v \in P_2$ then there is an uv -walk of length no more than $s + \ell_0$. Define $\ell_m = \min\{d(x_1, u) + d(y_1, v), d(x_2, u) + d(y_2, v)\}$. Since $d(x_1, u) + d(y_1, v) + d(x_2, u) + d(y_2, v) = 2\ell_0$, then $\ell_m \leq \ell_0$. We may assume that $\ell_m = d(x_1, u) + d(y_1, v)$. Then the uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $d(u, x_1)$, then moves to y_1 along the x_1y_1 -path of length $s - d(x_1, y_1)$, and finally moves to v along the y_1v -path of length $d(y_1, v)$, is an even uv -walk. Notice that $\ell_0(W_{uv}) = d(u, x_1) + s - d(x_1, y_1) + d(y_1, v)$. Since $d(x_1, u) + d(y_1, v) \leq \ell_0$ and $d(x_1, y_1) \geq 1$, then $\ell(W_{uv}) < s + \ell_0$. \square

Lemma 6. *Let $s \geq 5$ be an odd integer and G be an (s, ℓ_0) -two bar barbell. If ℓ_0 is even, then $k(G) = (s + \ell_0 - 1)/2$.*

Proof. Let v' be the median vertex on P_1 and let u' be the vertex such that $d(u', x_1) = d(v', x_1) - 1$. Then $d(v', x_1) = \ell_0/2$ and $d(u', v') = 1$. Notice that the shortest even $u'v'$ -walk $W_{u'v'}$ is the walk that starts at u' , moves to x_1 along the path of length $\ell_0/2 - 1$, then moves one time around the cycle C_1 and back at x_1 , then moves to u' along the path of length $\ell_0/2 - 1$, and finally moves to v' along the path of length 1. notice that $\ell_0(W_{u'v'}) = s + \ell_0 - 1$ and hence $k(G) \geq (s + \ell_0 - 1)/2$.

Let u and v be two vertices in G and we assume that $d(u, v)$ is odd. Considering the proof of Lemma 3 it remains to show that if $u \in P_1$ and $v \in P_2$, then there is an uv -walk of length no more than $s + \ell_0 - 1$. Define $\ell_m = \min\{d(x_1, u) + d(y_1, v), d(x_2, u) + d(y_2, v)\}$. Since $d(x_1, u) + d(y_1, v) + d(x_2, u) + d(y_2, v) = 2\ell_0$, then $\ell_m \leq \ell_0$. We may assume that $\ell_m = d(x_1, u) + d(y_1, v)$. Then the uv -walk W_{uv} that starts at u , moves to x_1 along the ux_1 -path of length $d(u, x_1)$, then moves to y_1 along the x_1y_1 -path of length $s - d(x_1, y_1)$, and finally moves to v along the y_1v -path of length $d(y_1, v)$ is an even uv -walk. Notice that $\ell(W_{uv}) = d(u, x_1) + s - d(x_1, y_1) + d(y_1, v)$. Since $d(x_1, u) + d(y_1, v) \leq \ell_0$ and $d(x_1, y_1) \geq 1$, then $\ell(W_{uv}) \leq s + \ell_0 - 1$. \square

Combining Lemma 5 and Lemma 6 we have the following result.

Theorem 7. *Let $s \geq 5$ be an odd integer and G be an (s, ℓ_0) -two bar barbell. Then $k(G) = \lceil (s + \ell_0 - 1)/2 \rceil$.*

Let C_1 be the cycle

$$v_1 - v_2 - v_3 - \cdots - v_s - v_1$$

and let C_2 be the cycle

$$v_{s+1} - v_{s+2} - v_{s+3} - \cdots - v_{2s} - v_{s+1}$$

each of length s . For $i = 1, 2, \dots, s$, we define the path P_i to be a path of length ℓ_0 connecting v_i and v_{s+i} . An (s, ℓ_0) -prism is a graph consisting of disjoint odd cycles C_1 and C_2 each of length s connected by paths P_i , $i = 1, 2, \dots, s$, each of length ℓ_0 . As a consequence and generalization of Theorem 7 we have the following theorem.

Theorem 8. *Let $s \geq 5$ be an odd integer. If G is an (s, ℓ_0) -prism. Then $k(G) = \lceil (s + \ell - 1)/2 \rceil$.*

Proof. The proof is similar to the proof of Theorem 7. □

References

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