IOP Conf. Series: Materials Science and Engineering **300** (2018) 012081 doi:10.1088/1757-899X/300/1/012081

Primitive graphs with small exponent and small scrambling index

Ita Y Surbakti, Saib Suwilo* Ericha A Butar-butar, and Helda Oktaviani

Department of Mathematics, Universitas Sumatera Utara, Medan 20155, Indonesia

E-mail: *saib@usu.ac.id

Abstract. A connected graph G is primitive provide there is a positive integer k such that for each pair of vertices u and v there is a uv-walk of length k. The smallest of such positive integer k is the exponent of G and is denoted by $\exp(G)$. The scrambling index of a primitive graph G, denoted by $k(G)$, is the smallest positive integer k such that for each pair of vertices u and v there is a vertex w such that there is a uw-walk and a vw-walk of length k. By an n-chainring $CR(n)$ we mean a graph obtained from an n-cycle by replacing each edge of the n-cycle by a triangle. By a (q,p)-dory, $D(q,p)$, we mean a graph with vertex set $V(D(q,p)) = V(P_q \times P_p) \cup \{w_1, w_2\}$ and edge set $E(D(q,p)) = E(P_q \times P_p) \cup \{w_1 - (u_i, v_1) : i = 1, 2, \ldots, q\} \cup \{w_2 - (u_i, v_p) : i = 1, 2, \ldots, q\}$, where P_n is a path on n vertices. We discus the exponent and scrambling index of an n-chainring and (q,p)-dory. We present formulae for exponent and scrambling index in terms of their diameter.

1. Introduction

Let G be a simple graph. We follow graph terminologies from [1,2]. Let u and v be two vertices in a graph G. A walk connecting u and v is denoted by W_{uv} or uv-walk. A uv-path is a uv-walk without repeated vertices except possibly $u = v$. A cycle is a closed path. The length of a walk W_{uv} is denoted by $\ell(W_{uv})$. By a triangle we mean a cycle of length three. A walk is even or odd if it is of even or odd length respectively. For a connected graph G the distance $d(u,v)$ of vertices u and v in G is defined to be the length of a shortest uv-path in G. The diameter of a connected graph G, denoted by $\mathrm{diam}(G)$, is defined to be

$$\mathrm{diam}(G) = \max_{u,v}\{d(u,v)\}.$$

A connected graph G is said to be primitive if there is a positive integer k such that for each pair of vertices u and v in G there is a uv-walk of length k. The exponent of a primitive graph G, denoted by $\exp(G)$, is the smallest of such positive integer k. The scrambling index of a primitive graph G is the smallest positive integer k such that for each pair of vertices u and v there is a uv-walk of length $2k$ [3–5]. It is known that a connected graph is primitive if and only if it contains an odd cycle [2]. From definition we have $\exp(G) \geq \mathrm{diam}(G)$. Notice also that if $\mathrm{diam}(G)$ of G is even then by definition $k(G) \geq \mathrm{diam}(G)$. If the $\mathrm{diam}(G)$ is odd, then there is a pair of vertices u and v such that the shortest even uv-walk is of length $\mathrm{diam}(G)+1$. Hence if $\mathrm{diam}(G)$ is odd, we have $k(G) \geq (\mathrm{diam}(G)+1)/2$. Therefore, for a primitive graph G we find that $k(G) \geq \left\lceil \frac{\mathrm{diam}(G)}{2} \right\rceil$. Therefore, for a primitive graph G with the smallest cycle of odd length $r \geq 3$

$$\mathrm{diam}(G) \leq \exp(G) \text{ and } \left\lceil \frac{\mathrm{diam}(G)}{2} \right\rceil \leq k(G). \tag{1}$$

In this paper we discuss two-classes of primitive graph with small exponent and small scrambling, that is, primitive graphs G with $\exp(G) = \mathrm{diam}(G)$ and $k(G) = \left\lceil \frac{\mathrm{diam}(G)}{2} \right\rceil$. For positif integer $n \geq 3$, an n-chainring is a graph obtained form an n-cycle by replacing each edge of the n-cycle by a triangle. More precisely, an n-chainring $CR(n)$ is a primitive graph on $2n$ vertices consisting of the the n-cycle $v_2 - v_4 - \cdots - v_{2n-2} - v_{2n} - v_2$ and the $2n$-cycle $v_1 - v_2 - v_3 - v_4 - \cdots - v_{2n-1} - v_{2n} - v_1$. The graph of $CR(8)$ is given in Figure 1.

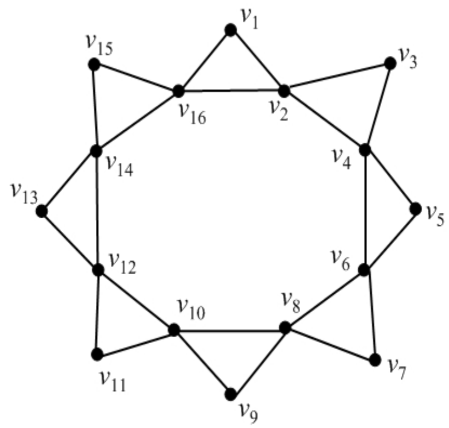

Figure 1. The Graph of $CR(8)$.

For positive integers p and q, let P_q be a path on q vertices $\{u_1, u_2, \ldots, u_q\}$ and P_p be a path on p vertices $\{v_1, v_2, \ldots, v_p\}$. By a (q,p)-dory, $D(q,p)$, we mean a graph with vertex set $V(D(q,p)) = V(P_q \times P_p) \cup \{w_1, w_2\}$ and edge set $E(D(q,p)) = E(P_q \times P_p) \cup \{w_1 - (u_i, v_1) : i = 1, 2, \ldots, q\} \cup \{w_2 - (u_i, v_p) : i = 1, 2, \ldots, q\}$.

In Section 2, we discuss properties of uv-walk especially uv-walk in an n-chainring. In Section 3, we discuss the exponent and scrambling index of n-chainring. In Section 4, we discuss the exponent and scrambling index of (q,p)-dory.

2. Properties of Walks

We discuss some properties of uv-walk necessary for our discussion.

Proposition 1. *Let G be a graph and let W_{uv} be a uv-walk of length $\ell(W_{uv})$ in G. If k is a positive integer such that $k \geq \ell(W_{uv})$ and $k \equiv \ell(W_{uv}) \bmod 2$, then there is a uv-walk of length k in G.*

Proof. Let

$$W_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i+1} - \cdots - v_m = v$$

be a uv-walk of length $\ell(W_{uv})$. Since $k \geq \ell(W_{uv})$ and $k \equiv \ell(W_{uv}) \bmod 2$, there is a nonnegative integer t such that $k - \ell(W_{uv}) = 2t$. Then the walk that starts at $u = v_0$, moves to v_{i+1} along the walk $u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i+1}$ and then moves t times around the closed walk $v_{i+1} - v_i - v_{i+1}$ and finally moves to v along the walk $v_{i+1} - \cdots - v_m = v$ is a uv-walk of length $k = \ell(W_{uv}) + 2t$. □

Proposition 2. *Let n be integer such that $n \geq 4$. Then for any pair of vertices u and v in $CR(n)$ there is a uv-path of length $d(u, v) + 1$.*

Proof. Let u and v be any two vertices in $CR(n)$ and let the path

$$P_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i+1} - \cdots - v_\ell = v$$

be a uv-path of length $d(u, v) = \ell$. Since every edge $v_i - v_{i+1}$ of the path P_{uv} lies on a triangle, then there is a vertex $v_{i'}$ such that the closed path $v_i - v_{i'} - v_{i+1} - v_i$ is a triangle. This implies the uv-path

$$P_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i'} - v_{i+1} - \cdots - v_\ell = v$$

is a uv-path of length $d(u, v) + 1$. □

3. Exponent and scrambling index of n-chainring

We discuss the exponent and scrambling index of $CR(n)$. We first present formulae for exponent and scrambling index in term of the diameter of $CR(n)$ and then present formulae that depends on n.

Theorem 3. *Let n be a positive integer with $n \geq 4$. Then* $\exp(CR(n)) = \operatorname{diam}(CR(n))$.

Proof. From (1) we have $\exp(CR(n)) \geq \operatorname{diam}(CR(n))$. It remains to show that $\exp(CR(n)) \leq \operatorname{diam}(CR(n))$. For each pair of vertices u and v we show that there exists a uv-walk of length $\operatorname{diam}(CR(n))$. Notice that for each pair of vertices u and v there is a uv-path P_{uv} of length $d(u, v)$. If $d(u, v) = \operatorname{diam}(CR(n)) \bmod 2$, then by Proposition 1 the path P_{uv} can be extended to a walk W_{uv} of length diam(CR(n)). If $d(u, v) \not\equiv \operatorname{diam}(CR(n)) \bmod 2$, then by Proposition 2 there is a path P'_{uv} of length $\ell(P'_{uv}) = d(u, v) + 1$. We now have $\ell(P'_{uv}) \equiv \operatorname{diam}(CR(n))$. Hence Proposition 1 guarantees that the path P'_{uv} can be extended to a walk W_{uv} of length $\operatorname{diam}(CR(n))$. □

Corollary 4. *Let n be a positive integer with $n \geq 4$. Then*

$$\exp(CR(n)) = \begin{cases} (n+1)/2, & \text{if } n \text{ is odd} \\ (n+2)/2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, the $\operatorname{diam}(G) = d(v_1, v_n) = (n+1)/2$. If n is even, the $\operatorname{diam}(G) = d(v_1, v_{n+1}) = (n+2)/2$. □

Theorem 5. *Let n be a positive integer such that $n \geq 3$. Then* $k(CR(n)) = \left\lceil \frac{\operatorname{diam}(CR(n))}{2} \right\rceil$.

Proof. From (1) we have $k(CR(n)) \geq \left\lceil \frac{diam(CR(n))}{2} \right\rceil$. It remains to show that $k(CR(n)) \leq \left\lceil \frac{diam(CR(n))}{2} \right\rceil$.

If the $\operatorname{diam}(CR(n))$ is even, then by Proposition 1 for each pair of vertices u and v there is a uv-walk of length $diam(CR(n))$. Thus we conclude that $k(CR(n)) \leq \operatorname{diam}(CR(n))/2$. If the $\operatorname{diam}(CR(n))$ is odd, then the shortest even walk connecting u_0 and v_0 is of length $\operatorname{diam}(CR(n)) + 1$. Notice that for every pair of vertices u and v, $d(u, v) \leq \operatorname{diam}(CR(n))$. Proposition 1 and Proposition 2 imply that for each pair of vertices u and v there is an even uv-walk of length $\operatorname{diam}(CR(n)) + 1$. Hence $k(CR(n)) \leq \frac{\operatorname{diam}(CR(n))+1}{2}$. We now conclude that $k(CR(n)) \leq \left\lceil \frac{\operatorname{diam}(CR(n))}{2} \right\rceil$.

We now conclude that $k(CR(n)) = \left\lceil \frac{diam(CR(n))}{2} \right\rceil$. □

Corollary 6. *For positive integer $n \geq 3$, $k(CR(n)) = \lfloor n/4 \rfloor + 1$.*

Proof. Suppose n is even. Then the $\mathrm{diam}(CR(n)) = (n+2)/2$ and is obtained by the v_1v_{n+1}-path $v_1 - v_2 - v_4 - \cdots - v_n - v_{n+1}$. If $n \equiv 0 \bmod 4$, then $\mathrm{diam}(CR(n)) = (4m+2)/2$ for some positive integer m. By Theorem 5 we have

$$\begin{aligned} k(CR(n)) &= \left\lceil \frac{\mathrm{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+2}{4} \right\rceil = \lceil m + 2/4 \rceil \\ &= m + 1 = \frac{n}{4} + 1. \end{aligned} \tag{2}$$

If $n \equiv 2 \bmod 4$, then $\mathrm{diam}(CR(n)) = (4m+4)/2$ for some positive integer m. By Theorem 5 we have

$$\begin{aligned} k(CR(n)) &= \left\lceil \frac{\mathrm{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+4}{4} \right\rceil = \lceil m + 1 \rceil \\ &= m + 1 = \frac{n-2}{4} + 1. \end{aligned} \tag{3}$$

Suppose now that n is odd. Then the $\mathrm{diam}(CR(n)) = (n+1)/2$ and is obtained by the v_1v_n-path $v_1 - v_2 - v_4 - \cdots - v_{n-1} - v_n$. If $n \equiv 1 \bmod 4$, then $\mathrm{diam}(CR(n)) = (4m+2)/2$ for some positive integer m. By Theorem 5 we have

$$\begin{aligned} k(CR(n)) &= \left\lceil \frac{\mathrm{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+2}{4} \right\rceil = \lceil m + 2/4 \rceil \\ &= m + 1 = \frac{n-1}{4} + 1. \end{aligned} \tag{4}$$

If $n \equiv 3 \bmod 4$, then $\mathrm{diam}(CR(n)) = (4m+4)/2$ for some positive integer m. By Theorem 5 we have

$$\begin{aligned} k(CR(n)) &= \left\lceil \frac{\mathrm{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+4}{4} \right\rceil = \lceil m + 1 \rceil \\ &= m + 1 = \frac{n-3}{4} + 1. \end{aligned} \tag{5}$$

From (2), (3), (4) and (5) we have $k(CR(n)) = \lfloor n/4 \rfloor + 1$. □

4. Exponent and scrambling index of (q, p)-dory

Let P_q be a path on q vertices $\{u_1, u_2, \ldots, u_q\}$ and P_p be a path on p vertices $\{v_1, v_2, \ldots, v_p\}$. By a (q, p)-dory, $D(q, p)$, we mean a graph with vertex set $V(D(q, p)) = V(P_q \times P_p) \cup \{w_1, w_2\}$ and edge set $E(D(q, p)) = E(P_q \times P_p) \cup \{w_1 - (u_i, v_1) : i = 1, 2, \ldots, q\} \cup \{w_2 - (u_i, v_p) : i = 1, 2, \ldots, q\}$. Since $D(q, p)$ is connected and contains triangles, $D(q, p)$ is primitive. We also note that if $p \geq q$, then $\mathrm{diam}(D(q, p)) = d(w_1, w_2) = p + 1$.

Theorem 7. *Let p and q be positive integers such that $p \geq q$. Then $\exp(D(q, p)) = p + 1$.*

Proof. We note from (1) that $\exp(D(q, p)) \geq \mathrm{diam}(D(q, p)) = p + 1$. It remains to show that $\exp(D(q, p)) \leq p + 1$. We show that for each pair of vertices x_0 and y_0 in $D(q, p))$ there is a walk connecting x_0 and y_0 of length $p + 1$.

If $d(x_0, y_0) \equiv p + 1 \bmod 2$, then by Proposition 1 there is a x_0, y_0-walk of length $p + 1$. It remains to consider the case where $d(x_0, y_0) \not\equiv p + 1 \bmod 2$.

Let P_{x_0,y_0} be the x_0, y_0-path of length $\ell(P_{x_0,y_0}) = d(x_0, y_0)$. If one end vertex of P_{x_0,y_0} is w_1 or w_2, then there is a path P_{x_0,y_0} such that $\ell(P_{x_0,y_0}) = d(x_0, y_0) + 1 \equiv p + 1$. We now assume that the end vertices of P_{x_0,y_0} are not w_1 or w_2. We consider three cases.

Case 1. The vertices $x_0 = (u_i, v_j)$ and $y_0 = (u_i, v_k)$ for some $1 \leq i \leq q$ and $1 \leq j < k \leq p$. We note that $d(x_0, y_0) = k - j \not\equiv p + 1 \bmod 2$. We assume without loss of generality that $p - k \geq j - 1$. The walk that starts at (u_i, v_j) moves to (u_i, v_1) along the path of length $(j-1)$, then moves one times around the triangle $(u_i, v_1) - w_1 - (u_{i-1}, v_1) - (u_i, v_1)$, and finally moves to (u_i, v_k) along the path of length $k-1$ is a x_0, y_0-walk of length $k+j+1$. We note that $k+j+1 = 2(j-1)+k-j+3$. Since $k-j \not\equiv p+1 \bmod 2$, we have $k+j+1 \equiv p+1 \bmod 2$. Moreover since $(j-1) \leq n-k$, we have $k+j+1 \leq p+2$. Since $k+j+1 \equiv p+1 \mod 2$, then $k+j+1 \leq p+1$.

Case 2. The vertices $x_0 = (u_i, v_k)$ and $y_0 = (u_j, v_k)$ for some $1 \leq k \leq p$ and $1 \leq i < j \leq q$. We note that $d(x_0, y_0) = j - i \not\equiv p + 1 \bmod 2$. Assume without loss of generality that $p - k > k - 1$. If $(j - i)$ is even, then the walk that starts at (u_i, v_k) moves to (u_i, v_1) along the path of length $k - 1$, then moves to (u_j, v_1) along the path $(u_i, v_1) - (u_{i+1}, v_1) - w_1 - (u_j, v_1)$ of length 3, and finally moves to (u_j, v_k) along the path of length $k - 1$, is a x_0, y_0-path of length $2(k - 1) + 3$. Since $2(k-1)+3 \not\equiv j-i \bmod 2$, we have $2(k-1)+3 \equiv p+1 \bmod 2$. We note that $n-k > k-1$. Therefore $2(k-1)+3 \leq p+2$. Since $2(k-1)+3 \equiv p+1 \bmod 2$, we conclude that $2(k-1)+3 \leq p+1$.

If $(j - i)$ is odd, then the walk that starts at (u_i, v_k) moves to (u_i, v_1) along the path of length $k-1$, then moves to (u_j, v_1) along the path $(u_i, v_1) - w_1 - (u_j, v_1)$ of length 2, and finally moves to (u_j, v_k) along the path of length $k - 1$, is a x_0, y_0-path of length $2(k - 1) + 2$. Since $2(k - 1) + 2 \not\equiv j - i \bmod 2$, we have $2(k - 1) + 2 \equiv p + 1 \bmod 2$. We note that $n - k > k - 1$. Therefore $2(k - 1) + 2 \leq p + 1$.

Case 3. The vertices $x_0 = (u_i, v_j)$ and $y_0 = (u_r, v_s)$ for some $1 \leq j < s \leq p$ and $1 \leq i < r \leq q$. Notice that $d(x_0, y_0) = (r - i) + (s - j) \not\equiv p + 1 \bmod 2$. Without loss of generality we assume that $j-1 \leq p-s$. If $(s-j) \equiv p+1 \bmod 2$, then the walk that starts at (u_i, v_j) moves to (u_i, v_1) along the path of length $(j - 1)$, then moves to (u_r, v_1) along the path $(u_i, v_1) - w_1 - (u_r, v_1)$, and finally moves to (u_r, u_s) along the path of length $(j - 1) + (s - j)$ is a x_0, y_0-path of length $2(j - 1) + 2 + (s - j) \equiv p + 1 \bmod 2$. Since $j - 1 \leq p - s$, then $2(j - 1) + 2 + (s - j) \leq p + 1$.

If $(s - j) \not\equiv p + 1 \bmod 2$, then the walk that starts at (u_i, v_j) moves to (u_i, v_1) along the path of length $(j-1)$, then moves to (u_r, v_1) along the path $(u_i, v_1) - (u_{i+1}, v_1) - w_1 - (u_r, v_1)$, and finally moves to (u_r, u_s) along the path of length $(j - 1) + (s - j)$ is a x_0, y_0-path of length $2(j - 1) + 3 + (s - j) \equiv p + 1 \bmod 2$. Since $j - 1 \leq p - s$, then $2(j - 1) + 3 + (s - j) \leq p + 2$. Since $2(j - 1) + 3 + (s - j) \equiv p + 1 \bmod 2$, we conclude that $2(j - 1) + 3 + (s - j) \leq p + 1$.

Therefore, for each pair of vertices x_0 and y_0 there is a x_0, y_0-walk W_{x_0,y_0} of length $\ell(W_{x_0,y_0}) \leq p + 1$. Proposition 1 guarantees that for each for each pair of vertices x_0 and y_0 there is a x_0, y_0-walk W_{x_0,y_0} of length $\ell(W_{x_0,y_0}) = p + 1$. Hence $\exp(D(q, p)) \leq p + 1$. □

Theorem 8. *Let p and q be positive integers such that $p \geq q$. Then* $k(D(q,p)) = \left\lceil \frac{p+1}{2} \right\rceil$.

Proof. From (1) we have $k(D(q,p)) \geq \left\lceil \frac{diam(D(q,p))}{2} = \frac{p+1}{2} \right\rceil$. It remains to show that $k(D(q,p)) \leq \left\lceil \frac{p+1}{2} \right\rceil$.

If $p+1$ is even, then by Proposition 1 for each pair of vertices x_0 and y_0 there is a x_0, y_0-walk of even length $p + 1$. Thus we conclude that $k(D(q,p)) \leq (p + 1)/2$.

If the $p + 1$ is odd, then the shortest even walk connecting w_1 and w_2 is of length $p + 2$. Notice that from Theorem 7 for every pair of vertices x_0 and y_0 there is a x_0y_0-walk of length $p + 1$. This implies for each each pair of vertices x_0 and y_0 there is a x_0y_0-walk of even length $p + 2$. Hence if $p + 1$ is odd, $k(D(q,p)) \leq (p + 2)/2$.

Therefore we now conclude that $k(D(q,p)) \leq \left\lceil \frac{p+1}{2} \right\rceil$. Hence we now have $k(D(q,p)) = \left\lceil \frac{p+1}{2} \right\rceil$. $\square$

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