

The grey Shapley value: an axiomatization

M Ekici^{1,*}, O Palanci² and S Z Alparslan Gök²

¹Usak University, Faculty of education, Department of Mathematics and Science Education, 61 200 Usak, Turkey

²Süleyman Demirel University, Faculty of Arts and Sciences, Department of Mathematics, 32 260 Isparta, Turkey

E-mail: *mutafa.ekici@usak.edu.tr

Abstract. This study focuses on an interesting class of cooperative games where the coalitional values are interval grey numbers. These cooperative games are called cooperative grey games. In this paper, we deal with an axiomatization of the grey Shapley value. We introduce the Banzhaf value and the egalitarian rule by using cooperative grey game theory. Finally, we conclude the paper with a conclusion.

1. Introduction

In this paper, we consider the cooperative games by using grey calculus. The grey systems theory is a new methodology that examines the study of problems involving small samples and poor information. In grey system theory, random variables are regarded as grey numbers, and a stochastic process is referred to as grey process. A grey system is defined as a system containing information presented as grey numbers; and a grey decision is defined as a decision made within a grey system [4,5,13].

This paper focuses on cooperative grey game is where interval grey numbers are taken. This work constructs the framework system of cooperative game theory based on grey data and applies cooperative grey game theory. With the application theory of a grey system, this paper deals with cooperative game issues based on incomplete information and grey uncertainty.

The article is organized as follows. We give basic notions from the theory of grey calculus is Section 2. The grey Shapley value is studied on the class of size monotonic grey games and axiomatic characterization of the grey Shapley value is given in Section 3. In Section 4, we introduce the grey Banzhaf value and the grey egalitarian rule on the special classes of cooperative grey games.

2. Preliminaries

In this section some preliminaries from grey calculus, and grey calculus used for cooperative grey games are given [7,11].

A grey number is such a number whose exact value is known but a range within that the value lies is known. In applications, a grey number in general is a grey or a general set of numbers.

In this paper, we consider the interval grey numbers.

A grey number with both a lower limit $\underline{a}$ and an upper limit $\bar{a}$ is called a grey number, denoted as $\otimes \in [\underline{a}, \bar{a}]$.



For example, the weight of a seal is between 30 and 35 kg. A specific person's height is between 1.9 and 2.0 meters. These two grey numbers can be respectively written as

$$\otimes_1 \in [30, 35] \text{ and } \otimes_2 \in [1.9, 2.0].$$

Now, we discuss various operations on grey grey numbers.

Let

$$\otimes_1 \in [a, b], a < b \text{ and } \otimes_2 \in [c, d], c < d.$$

The sum of $\otimes_1$ and $\otimes_2$, written $\otimes_1 + \otimes_2$, is defined as follows:

$$\otimes_1 + \otimes_2 \in [a + c, b + d].$$

For example, let $\otimes_1 \in [2, 7]$ and $\otimes_2 \in [3, 6]$, then,

$$\otimes_1 + \otimes_2 \in [5, 13].$$

Assume that $\otimes \in [a, b], a < b$ and k is a positive real number. The scalar multiplication of k and $\otimes$ is defined as follows:

$$k\otimes \in [ka, kb].$$

We denote by $\mathcal{G}(\mathbb{R})$ the set of grey grey numbers in $\mathbb{R}$. Let $\otimes_1, \otimes_2 \in \mathcal{G}(\mathbb{R})$ with $\otimes_1 \in [a, b], a < b; \otimes_2 \in [c, d], c < d, |\otimes_1| = b - a$ and $\alpha \in \mathbb{R}_+$. Then,

- (i) $\otimes_1 + \otimes_2 \in [a + c, b + d];$
- (ii) $\alpha\otimes = [\alpha a, \alpha b].$

By (i) and (ii) we see that $\mathcal{G}(\mathbb{R})$ has a cone structure.

In general, the difference of $\otimes_1$ and $\otimes_2$ is defined as follows:

$$\otimes_1 \ominus \otimes_2 = \otimes_1 + (-\otimes_2) \in [a - d, b - c],$$

(see[8]).

For example, let $\otimes_1 \in [6, 8]$ and $\otimes_2 \in [2, 5]$, then we have

$$\otimes_1 \ominus \otimes_2 \in [6 - 5, 8 - 2] = [1, 6],$$

$$\otimes_2 \ominus \otimes_1 \in [2 - 8, 5 - 6] = [-6, -1].$$

Different from the above subtraction we use a partial subtraction operator. We define $\otimes_1 - \otimes_2$, only if $|b - a| \geq |d - c|$, by $\otimes_1 - \otimes_2 \in [a - c, b - d]$. Note that $a - c \leq b - d$. We recall that $[a, b]$ is weakly better than $[c, d]$, which denote by $[a, b] \succeq [c, d]$, if and only if $a \geq c$ and $b \geq d$. We also use the reverse notation $[a, b] \preceq [c, d]$ if and only if $a \leq c$ and $b \leq d$ (for details see [1,3]).

Notice that if we make a comparison with the above example, then in our case $[6, 8] - [2, 5]$ is not defined. But, $[2, 5] - [6, 8]$ is defined.

Let $\otimes_1 \in [2, 5]$, and $\otimes_2 \in [6, 8]$, $\otimes_1 - \otimes_2$ is defined since $|5 - 2| \geq |8 - 6|$, but $\otimes_2 - \otimes_1$ is not defined since $|8 - 6| = 2 \not\geq 3 = |5 - 2|$, then we have

$$\otimes_1 - \otimes_2 \in [2 - 6, 5 - 8] = [-4, -3]$$

A cooperative grey game is an ordered pair $\langle N, w \rangle$ where $N = \{1, \dots, n\}$ is the set of players, and $w = \otimes : 2^N \rightarrow \mathcal{G}(\mathbb{R})$ is characteristic function such that $w(\emptyset) = \otimes_\emptyset \in [0, 0]$, grey payoff function $w(S) = \otimes_S \in [\underline{A}_S, \overline{A}_S]$ refers to the value of the grey expectation benefit belonging to coalition $S \in 2^N$, where $\overline{A}^S$ and $\underline{A}_S$ represent the maximum and minimum possible profits of

the coalition S . So, a cooperative grey game can be considered as a classical cooperative game with grey profits $\otimes$. Grey solutions are useful to solve reward/cost sharing problems with grey data using cooperative grey game as a tool. Building blocks for grey solutions are grey payoff vectors, i.e., vectors whose components belong to $\mathcal{G}(\mathbb{R})$ we denote by $(\mathcal{G}(\mathbb{R}))^N$ the set of all such grey payoff vectors. We denote by $\mathcal{GG}^N$ the family of all cooperative grey games.

We call a game $\langle N, w \rangle$ *grey size monotonic* if $\langle N, |w| \rangle$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMGG^N$ the class of grey size monotonic games with player set N . The grey marginal operators and the grey Shapley value are defined on $SMGG^N$. Denote by $\Pi(N)$ the set of permutations $\sigma : N \rightarrow N$ of N . The *grey marginal operator* $m^\sigma : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ corresponding to σ , associates with each $w' \in SMGG^N$ the *grey marginal vector* $m^\sigma(w')$ of w' with respect to σ defined by

$$m_i^\sigma(w') := w'(P^\sigma(i) \cup \{i\}) - w'(P^\sigma(i)) \in [\underline{A_{P^\sigma(i) \cup \{i\}}} - \underline{A_{P^\sigma(i)}}], \overline{A_{P^\sigma(i) \cup \{i\}}} - \overline{A_{P^\sigma(i)}}],$$

for each $i \in N$, where

$$P^\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\},$$

and σ^{-1} denotes the entrance number of player i . For grey size monotonic games $\langle N, w \rangle$, $w(T) - w(S) \in w(T) - w(S)$ is defined for all $S, T \in 2^N$ with $S \subset T$ since $|w(T)| = |w|(T) \geq |w|(S) = |w(S)|$. We notice that for each $w' \in SMGG^N$ the grey marginal vectors $m^\sigma(w)$ are defined for each $\sigma \in \Pi(N)$, because the monotonicity of $|w|$ implies $\overline{A_{S \cup \{i\}}} - \underline{A_{S \cup \{i\}}} \geq \overline{A_S} - \underline{A_S}$, which can be written as $\overline{A_{S \cup \{i\}}} - \overline{A_S} \geq \underline{A_{S \cup \{i\}}} - \underline{A_S}$. So, $w'(S \cup \{i\}) - w'(S) \in w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \notin S$. Next, we notice that all the grey marginal vectors of a grey size monotonic game are efficient grey payoff vectors.

The *grey Shapley value* $\Phi : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ is defined by [9]

$$\Phi'(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w) \in \left[\frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(\underline{A}), \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(\overline{A}) \right]. \quad (1)$$

Next, we recall that efficiency, null player and additivity axioms for solutions $f : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$.

Efficiency (E): For every $w \in SMGG^N$ it holds that $\sum_{i \in N} f_i(w) = w(N)$.

Let $w \in SMGG^N$ and $i \in N$. Then, i is called a *null player* if $w(S \cup \{i\}) = w(S)$, for each $S \in 2^{N \setminus \{i\}}$.

Null Player (NP): If $i \in N$ is a null player in a grey game $w \in SMGG^N$, then $f_i(w) \in [0, 0]$

Additivity (A): For every $w, v \in SMGG^N$ it holds that $f(w + v) = f(w) + f(v)$.

Now, we mention the fairness property. Fairness property is introduced in van den Brink (2002).

Let $w \in SMGG^N$ and $i, j \in N$. Then, i and j are called *symmetric players*, if $w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S)$, for each S with $i, j \notin S$.

Symmetry (S): If $i, j \in N$ are symmetric players in $w \in SMGG^N$, then $f_i(w) = f_j(w)$.

Fairness state that if to an grey game $w \in SMGG^N$ we add an grey game $v \in SMGG^N$ in which players i and j are symmetric, then the grey payoffs of players i and j change by the same amount.

Fairness (F): If $i, j \in N$ are symmetric players in $w \in SMGG^N$, then $f_i(w + v) - f_i(w) = f_j(w + v) - f_j(w)$ for all $v \in SMGG^N$ [12].

3. An axiomatic characterization of grey Shapley value

In this section, we give the characterization of grey Shapley value by using fairness property on the special subclass of cooperative grey games. Firstly, we mention about the relationship between our axioms.

Every grey solution that fulfills symmetry and additivity also satisfies fairness. Further, every grey solution that fulfills the null player property and fairness also satisfies symmetry. These propositions are an extension of the propositions obtained by [12] to cooperative grey games.

Proposition 1. *If $f : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ fulfills symmetry and additivity, then f also satisfies fairness.*

Proof. We know that $f : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ fulfill symmetry and additivity. Let $f : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ satisfy symmetry and additivity. If $i, j \in N$ are symmetric in $w \in SMGG^N$, then for every $v \in SMGG^N$ it holds that

$$\begin{aligned} f_i(w + v) - f_i(w) &= f_i(w) + f_i(v) - f_i(w) && \text{(from A)} \\ &= f_i(v) && \text{(from S)} \\ &= f_j(v) \\ &= f_j(v) + f_j(w) - f_j(w) && \text{(from A)} \\ &= f_j(w + v) - f_j(w). \end{aligned}$$

Thus fulfills fairness. □

Proposition 2. *If $f : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ fulfills the null player property and fairness, then f also satisfies symmetry.*

Proof. Let $f : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ fulfill the null player property and fairness. For the null game $w_0 \in SMGG^N$ given by $w_0(S) = [0, 0]$ for all $S \subset N$, the null player property implies that $f_i(w_0) = [0, 0]$ for all $i \in N$. If $i, j \in N$ are symmetric players in $w \in SMGG^N$, then

$$\begin{aligned} f_i(w) &= f_i(w_0 + w) - f_i(w_0) && \text{(from F)} \\ &= f_j(w_0 + w) - f_j(w_0) \\ &= f_j(w) \end{aligned}$$

Thus, f fulfills symmetry. □

We know that the grey Shapley value is characterized by efficiency, the null player property, symmetry and additivity. By Proposition 1 it also guarantees fairness. Now, we give the main result of this paper.

Theorem 3. *A grey solution $g : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ is equal to the grey Shapley value if and only if it fulfills efficiency, the null player property and fairness.*

Proof. The proof is a straight forward generalization from the classical case and can be obtained by following the steps of Theorem 2.5 in [12]). □

Logical independence of the three axioms of Theorem 3 can be illustrated by the following two well-known solutions. Grey Banzhaf value fulfills the null player property and fairness but it does not satisfy efficiency. Grey egalitarian rule fulfills efficiency and fairness but it does not satisfy the null player property.

4. The grey Banzhaf value and the grey egalitarian rule

A cooperative game describes a situation in which a finite set of n players can generate certain payoffs by cooperation. A one-point solution concept for cooperative games is a function which assigns to every cooperative game a n -dimensional real vector which represents a payoff distribution over the players. The study of solution concepts is central in cooperative game theory. Two well-known solution concepts are the Shapley value as proposed by [10], and the Banzhaf value, initially introduced in the context of voting games by [2]. Now, we introduce the Banzhaf value by using grey uncertainty.

The classical Banzhaf value $\beta : G^N \rightarrow \mathbb{R}^N$ given by

$$\beta_i(v) := \frac{1}{2^{|N|-1}} \sum_{i \in S} (v(S) - v(S \setminus \{i\}))$$

for all $i \in N$.

The Banzhaf value considers that every player is equally likely to enter to any coalition whereas the Shapley value assumes that every player is equally likely to join to any coalition of the same size and all coalitions with the same size are equally likely. In addition, the Shapley value is efficient, while the Banzhaf value is not efficient. Thus, the Shapley value distributes the total utility among players while the total amount that players get from Banzhaf's allocation depends on the structure of the TU game.

The grey Banzhaf value is defined for $SMGG^N$ since the grey marginal operators is defined for $SMGG^N$.

The grey Banzhaf value $\beta : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$ is defined by

$$\beta'(w) \in \left[\frac{1}{2^{|N|-1}} \sum_{i \in S} (\underline{w}(S) - \underline{w}(S \setminus \{i\})), \frac{1}{2^{|N|-1}} \sum_{i \in S} (\overline{w}(S) - \overline{w}(S \setminus \{i\})) \right],$$

for each $w \in SMGG^N$.

Example 4. Let $\langle N, w \rangle$ be a cooperative grey game with $N = \{1, 2, 3\}$ and $w(1) = w(13) \in [7, 7]$, $w(12) \in [12, 17]$, $w(123) \in [24, 29]$, and $w(S) \in [0, 0]$ otherwise. The grey Shapley value of this game can be calculated as follows: Then the grey marginal vectors are given in the following table, where $\sigma : N \rightarrow N$ is identified with $(\sigma(1), \sigma(2), \sigma(3))$. Firstly, for $\sigma_1 = (1, 2, 3)$, we calculate the grey marginal vectors. Then,

$$\begin{aligned} m_1^{\sigma_1}(w) &= w(1) \in [7, 7], \\ m_2^{\sigma_1}(w) &= w(12) - w(1) \in [5, 10], \\ m_3^{\sigma_1}(w) &= w(123) - w(12) \in [12, 12]. \end{aligned}$$

The others can be calculated similarly, which is shown in Table 1.

Table 1 illustrates the grey marginal vectors of the cooperative grey game in Example 4. The average of the six grey marginal vectors is the grey Shapley value of this game which can be shown as:

$$\Phi'(w) \in ([\frac{27}{2}, 16], [\frac{13}{2}, 9], [4, 4]).$$

The grey Banzhaf value of this game can be calculated as follows: The grey Banzhaf value is defined as

$$\beta'(w) \in \left[\frac{1}{2^{|N|-1}} \sum_{i \in S} (\underline{w}(S) - \underline{w}(S \setminus \{i\})), \frac{1}{2^{|N|-1}} \sum_{i \in S} (\overline{w}(S) - \overline{w}(S \setminus \{i\})) \right].$$

Table 1. Grey marginal vectors

σ	$m_1^\sigma(w')$	$m_2^\sigma(w')$	$m_3^\sigma(w')$
$\sigma_1 = (1, 2, 3)$	$m_1^{\sigma_1}(w') \in [7, 7]$	$m_2^{\sigma_1}(w') \in [5, 10]$	$m_3^{\sigma_1}(w') \in [12, 12]$
$\sigma_2 = (1, 3, 2)$	$m_1^{\sigma_2}(w') \in [7, 7]$	$m_2^{\sigma_2}(w') \in [17, 22]$	$m_3^{\sigma_2}(w') \in [0, 0]$
$\sigma_3 = (2, 1, 3)$	$m_1^{\sigma_3}(w') \in [12, 17]$	$m_2^{\sigma_3}(w') \in [0, 0]$	$m_3^{\sigma_3}(w') \in [12, 12]$
$\sigma_4 = (2, 3, 1)$	$m_1^{\sigma_4}(w') \in [24, 29]$	$m_2^{\sigma_4}(w') \in [0, 0]$	$m_3^{\sigma_4}(w') \in [0, 0]$
$\sigma_5 = (3, 1, 2)$	$m_1^{\sigma_5}(w') \in [7, 7]$	$m_2^{\sigma_5}(w') \in [17, 22]$	$m_3^{\sigma_5}(w') \in [0, 0]$
$\sigma_6 = (3, 2, 1)$	$m_1^{\sigma_6}(w') \in [24, 29]$	$m_2^{\sigma_6}(w') \in [0, 0]$	$m_3^{\sigma_6}(w') \in [0, 0]$

Then,

$$\begin{aligned}\beta_1(w) &= \frac{1}{2^2} \sum_{1 \in S} (w(S) - w(S \setminus \{1\})) \\ &\in [12\frac{1}{2}, 15].\end{aligned}$$

The others can be calculated similarly:

$$\beta_2(w) \in [5\frac{1}{2}, 8], \quad \beta_3(w) \in [3, 3].$$

Then, the grey Banzhaf value can be shown as

$$\beta'(w) \in ([12\frac{1}{2}, 15], [5\frac{1}{2}, 8], [3, 3]).$$

Remark 5. The grey Banzhaf value satisfies the null player property, and fairness but does not fulfill efficiency.

In Example 4, the grey Banzhaf value does not satisfy efficiency.

$$\begin{aligned}\sum_{i=1}^3 \beta'_i(w) &\in \beta'_1(w) + \beta'_2(w) + \beta'_3(w) \\ &= [12\frac{1}{2}, 15] + [5\frac{1}{2}, 8] + [3, 3] \\ &= [21, 26] \neq [24, 29] = w(N)\end{aligned}$$

A family of monotonic solutions to general cooperative games (coalitional form games where utility is not assumed to be transferable) is the egalitarian rule. The egalitarian rule is introduced by [6].

The classical egalitarian rule $\gamma : G^N \rightarrow \mathbb{R}^N$ is given by

$$\gamma_i(v) := \frac{v(N)}{|N|},$$

for all $i \in N$.

The grey egalitarian rule is defined for $\mathcal{G}G^N$. The grey egalitarian rule $\gamma' : \mathcal{G}G^N \rightarrow \mathcal{G}(\mathbb{R})^N$ is defined by

$$\gamma'(w) = \frac{w(N)}{|N|}, \text{ for each } w \in \mathcal{G}G^N.$$

Example 6. Consider again the cooperative grey game in Example 1. Then, the grey egalitarian rule of this game is:

$$\gamma'_i(w) = \frac{w(N)}{|N|} \in \frac{[24, 29]}{3} = [8, 9\frac{2}{3}], \quad i = 1, 2, 3.$$

Remark 7. The grey egalitarian rule satisfies efficiency and fairness but does not fulfill the null player property.

Example 8. Consider the cooperative grey game $\langle N, w \rangle$ with $N = \{1, 2, 3\}$ and $w(2) = w(12) \in [7, 7]$, $w(3) = w(13) \in [12, 17]$, $w(23) = w(N) \in [24, 29]$, and $w(S) \in [0, 0]$ otherwise. The grey egalitarian rule of this game is:

$$\gamma'_i(w) = \frac{w(N)}{|N|} \in \frac{[24, 29]}{3} = [8, 9\frac{2}{3}], \quad i = 1, 2, 3.$$

Here, the null player is player 1 and $\gamma'_1(w) \in [8, 9\frac{2}{3}]$. Thus, the grey egalitarian rule does not satisfy the null player property.

5. Conclusion

The Shapley value and the Banzhaf value are interesting one-point solutions in cooperative game theory. In this context, we give the characterization of the grey Shapley value by using fairness property. In addition the interval Banzhaf value and the egalitarian rule are introduced by using grey systems theory.

The grey Shapley value is axiomatically characterized but the grey Banzhaf value and the grey egalitarian rule are not axiomatically characterized. There exists a gap to be filled by characterizing the grey Banzhaf value and the grey egalitarian rule. It is possible to characterize the grey Banzhaf value and the grey egalitarian rule in the future studies.

References

- [1] Alparslan Gök S Z, Branzei O, Branzei R, and Tijs S 2011 Set-valued solution concepts using interval-type payoffs for interval games *Journal of Mathematical Economics* **47** 621–626.
- [2] Banzhaf J F 1965 Weighted Voting Doesn't Work: A Mathematical Analysis *Rutgers Law Review* **19** 317–343.
- [3] Branzei R, Dimitrov D and Tijs S 2003 Shapley-like values for interval bankruptcy games, *Economics Bulletin* **3** 1–8.
- [4] Deng J 1982) Control problems of Grey Systems *Systems and Control Letters* **5** 288–294.
- [5] Deng J 1985 *Grey System Fundamental Method* (Huazhong University of Science and Technology: Wuhan, China).
- [6] Kalai E and Samet D 1985 Monotonic Solutions to General Cooperative Games, *Econometrica* **53** (2) 307–327.
- [7] Liu S and Lin Y 2006 *Grey Information: Theory and Practical Applications* (Springer: Germany).
- [8] Moore R 1979 *Methods and Applications of Interval Analysis*, (SIAM Studies in Applied Mathematics: Philadelphia).
- [9] Palanci O, Alparslan Gök S Z, Ergün S, and Weber G W, 2015 Cooperative grey games and the grey Shapley value, *Optimization: A Journal of Mathematical Programming and Operations Research* **64** (8) 1657–1668.
- [10] Shapley L S 1953 A value for n -person games, *Annals of Mathematics Studies* **28** 307–317.
- [11] Tijs S 2003 *Introduction to Game Theory* Vol. 23 of Texts and Readings in Mathematics (Hindustan Book Agency: New Delhi, India).
- [12] van den Brink R 2002 An axiomatization of the Shapley value using a fairness property, *International Journal of Game Theory* **30** (3) 309–319.
- [13] Zhang J J, Wu D S, and Olson D L 2005 The method of grey related analysis to multiple attribute decision making problems with interval numbers, *Mathematical and Computer Modelling* **42** (9-10) 991–998.