IOP Conf. Series: Materials Science and Engineering **300** (2018) 012079 doi:10.1088/1757-899X/300/1/012079

Exponent and scrambling index of double alternate circular snake graphs

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Abstract. A graph is primitive if it contains a cycle of odd length. The exponent of a primitive graph G, denoted by $\exp(G)$, is the smallest positive integer k such that for each pair of vertices u and v in G there is a uv-walk length k. The scrambling index of a primitive graph G, denoted by $k(G)$, is the smallest positive integer k such that for each pair of vertices u and v in G there is a uv-walk of length $2k$. For an even positive integer n and an odd positive integer r, a (n, r)-double alternate circular snake graph, denoted by $DA(C_{r,n})$, is a graph obtained from a path $u_1u_2 \dots u_n$ by replacing each edge of the form $u_{2i}u_{2i+1}$ by two different r-cycles. We study the exponent and scrambling index of $DA(C_{r,n})$ and show that $\exp(DA(C_{r,n})) = n + r - 4$ and $k(DA(C_{r,n})) = (n + r - 3)/2$.

1. Introduction

Let $G(V, E)$ denote a graph on n vertices. Let u and v be any vertices in G, a *uv-walk* W_{uv} of length t connecting u and v is a sequence of vertices $u = u_0, u_1, u_2, \dots, u_t = v$ and a sequence of edges $\{u_0, u_1\}, \{u_1, u_2\}, \dots, \{u_{t-1}, u_t\}$, where the vertices and the edges are not necessarily distinct. The length of the walk W_{uv} is denoted by $\ell(W_{uv})$. A walk connecting u and v is closed whenever $u = v$, and is open otherwise. A *path* P_{uv} connecting u and v is a walk W_{uv} with distinct vertices, except possibly $u = v$. A *cycle* is a closed path. A walk W_{uv} with sequence of edges $\{u = u_0, u_1\}, \{u_1, u_2\}, \dots, \{u_{t-1}, u_t = v\}$ is also denoted by $W_{uv} : u = u_0 - u_1 - u_2 - \cdots - u_{t-1} - u_t = v$. The distance between vertices u and v, denoted by $d(u, v)$, is the length of a shortest path connecting u and v.

A graph G is connected provided that for each pairs of vertices u and v in G there is a walk connecting u and v. A connected graph G is primitive provided there is a positive integer m such that for each pair of vertices u and v in G, there is a uv-walk of length m. The smallest of such positive integer m is the exponent of G and is denoted by $\exp(G)$. It is well known that a graph G is primitive if and only if G has a cycle of odd length [3].

Alkelbek and Kirkland [1, 2] introduced the notion of scrambling index of primitive graph G for the first time in 2009. The scrambling index of a primitive graph G, denoted by $k(G)$, is the smallest positive integer k such that for any pair of distinct vertices u and v in G there exists a vertex w such that there is a uv-walk of length m and a vw-walk of length m. Chen and Liu [4] discussed the scrambling index of classes of primitive graph with the smallest cycle of length $s \geq 3$ and classes of primitive graphs with loops.

By an r-double circular DC_r we mean a connected graph consisting of two r-cycles that have one edge in common. The edge in common of the two r-cycles is called the base of the r-double circular or the the base of the r-cycles. Figure 1 presents a DC_5 with base is the edge $\{u_3, u_4\}$.

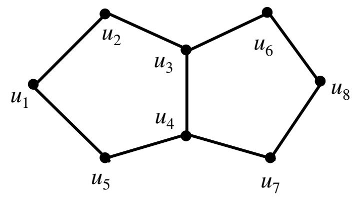

Figure 1. The graph of $DC(5)$

Let n be an even positive integer and r be an odd positive integer. A double alternate circular snake graph ($DA(C_{r,n})$ is a graph obtained from a path $u_1, u_2, \ldots, u_n$ by replacing each edges of the form $\{u_{2i}, u_{2i+1}\}$, $1 \leq i \leq \frac{n-2}{2}$ by an r-double circular with base $\{u_{2i}, u_{2i+1}\}$. Figure 2 presents a $DA(C_{5,6})$.

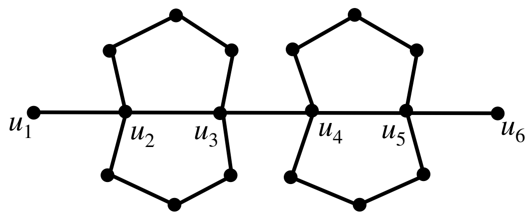

Figure 2. The graph of $DA(C_{5,6})$

Double alternate circular snake graph, especially double alternate triangular snake graph, has attracted many researchers [5, 6]. In this paper, we discussed the exponent and the scrambling index of double alternate circular snake graph, $DA(C_{r,n})$. We present formulae for exponent and scrambling index that depends on n and r.

2. Preliminaries

We discuss some properties of uv-walk in a graph and property of scrambling index.

Proposition 1. *Let G be a graph and let k and ℓ be positive integers such that $\ell \leq k$ and $k \equiv \ell \bmod 2$. If there is a uv-walk of length ℓ in G, then there is a uv-walk of length k in G.*

Proof. Since $k \geq \ell$ and $k \equiv \ell \bmod 2$, there is a nonnegative integer t such that $k = \ell + 2t$. Let W_{uv} be a uv-walk of length ℓ in G and let $\{v, w\}$ be an edge in W_{uv}. Then the uv-walk obtained by moving from u to v along the walk W_{uv} and then moves t times around the closed walk $v - w - v$ is a uv-walk of length k. □

Proposition 2. *Let G be a primitive graph and k' be an even positive integer. If for each pair of vertices u and v in G there is an even uv-walk W_{uv} of length $\ell(W_{uv}) \leq k'$, then $k(G) \leq k'/2$.*

Proof. For each pair of vertices u and v let W_{uv} be an even uv-walk of length $\ell(W_{uv}) \leq k'$. Since $k' \equiv \ell(W_{uv}) \bmod 2$, Proposition 1 implies there is a uv-walk of length k'. Since k' is even, then there exists a vertex w with the property that there is a uw-walk of length $k'/2$ and there is a vw-walk of length $k'/2$. Thus, $k(G) \leq k'/2$. □

For a pair of distinct vertices u and v in G the local scrambling index of u and v is the number

$$k_{u,v}(G) = \min_{w \in V(G)} \{k : \text{there are } uw\text{-walk and } vw\text{-walk of length } k\}.$$

We note that if the local scrambling index of u and v is $k_{u,v}(G)$, then by Proposition 1 for any positive integer $k' \geq k_{u,v}(G)$ we can find a vertex w such that there is a walk $u \overset{k'}{\leftrightarrow} w$ and $v \overset{k'}{\leftrightarrow} w$. This implies

$$k(G) = \max\{k_{u,v}(G) | u, v \in V(G), u \neq v\}.$$

3. Result

We discuss the exponent and scrambling index of double alternate circular snake graph $DA(C_{r,n})$ obtained from a path $u_1, u_2, \ldots u_n$ by replacing each edge of the form $\{u_{2i}, u_{2i+1}\}$, $1 \leq i \leq \frac{n-2}{2}$ by r-double cycle. We note that there are $\frac{n-2}{2}$ double r-cycles. Let P_n be the path $u_1 - u_2 - \cdots - u_n$ of length $n-1$ and let C_r be an r-cycle. For any vertices u_i and u_j in C_r, we define $P_{u_i u_j}$ to be the shortest path connecting u_i and u_j and $P'_{u_i u_j}$ to be the path connecting u_i and u_j of length $r - d(u_i, u_j)$. We note that since r is odd, $\ell(P_{u_i,u_j}) \not\equiv \ell(P'_{u_i,u_j}) \bmod 2$. We first present the result on the exponent of primitive double alternate circular snake head.

Theorem 3. *Let $n \geq 4$ is an even positive integer and $r \geq 3$ is an odd positive integer. Then* $\exp(DA(C_{r,n})) = n + r - 4$.

Proof. We first show that $\exp(DA(C_{r,n})) \geq n+r-4$. We note that for any pair of distinct vertices u and v in $DA(C_{r,n})$ we have $\exp(DA(C_{r,n})) \geq d(u, v)$. Hence $\exp(DA(C_{r,n})) \geq d(u_1, u_n) = n-1$. Notice that the shortest $u_1 u_{n-1}$-walk of length at least $n-1$ is the $u_1 u_{n-1}$-walk of length $n + r - 4$. Hence

$$\exp(DA(C_{r,n})) \geq n + r - 4.$$

We now show that $\exp(DA(C_{r,n})) \leq n + r - 4$. For each pair of vertices u_α and u_β it suffices to show that there is a $u_\alpha u_\beta$-walk of length $n + r - 4$. We note if $r \geq 5$, then for any pair of vertices u_α and u_β in $DA(C_{r,n})$ the distance $d(u_\alpha, u_\beta) \leq n + r - 6$. If $d(u_\alpha, u_\beta)$ is odd, then we can extend the path $P_{u_\alpha u_\beta}$ of length $d(u_\alpha, u_\beta)$ into a walk $W_{u_\alpha u_\beta}$ of length exactly $n + r - 4$.

We now assume that $d(u_\alpha, u_\beta)$ is even. Then $d(u_\alpha, u_\beta) < n + r - 4$. We consider four cases depending on the position of the vertices u_α and u_β.

Case 1. *The vertices u_α and u_β lie on the path P_n.* Assume that $\alpha < \beta$. Since $d(u_\alpha, u_\beta)$ is even, we have $d(u_\alpha, u_\beta) \leq n - 2$ and the path P_{u_α,u_β} must contains an edge of the form $\{u_{2i}, u_{2i+1}\}$ for some i. This implies the path that start at u_α moves to u_{2i} along the path of length $d(u_\alpha, u_{2i})$, then moves to v_{2i+1} along the path $P'_{u_{2i},u_{2i+1}}$ of length $r - 1$, then finally moves to u_β along the path of length $d(u_{2i+1}, u_\beta)$ is a $u_\alpha u_\beta$-path of odd length $\ell(P_{u_\alpha,u_\beta}) = d(u_\alpha, u_{2i}) + r - 1 + d(u_{2i+1}, u_\beta) \leq n - 3 + r - 1 = n + r - 4$.

Case 2. *The vertices u_α and u_β lie on a double circular $C_{2,r}$.* If u_α and u_β lie on the same cycle C_r, then there is an odd path P'_{u_α,u_β} of length $r - d(u_\alpha, u_\beta) \leq n + r - 4$. Suppose now u_α and u_β lie on two different cycles C_r. Since $d(u_\alpha, u_\beta)$ is even, we have $d(u_\alpha, u_\beta) \leq r - 1$. We note that the path P_{u_α,u_β} of length $d(u_\alpha, u_\beta)$ can be decomposed into two paths P_{u_α,u_y} and P_{u_y,u_β} for $y = 2i, 2i+1$. Since $d(u_\alpha, u_\beta)$ is even, then $\ell(P_{u_\alpha,u_y}) \equiv \ell(P_{u_y,u_\beta}) \bmod 2$. Assume that $\ell(P_{u_\alpha,u_y}) \leq \ell(P_{u_y,u_\beta})$. Then the walk that starts at u_α moves to u_y along the path P_{u_α,u_y}, then moves to u_β along the path P'_{u_y,u_β} of length $r - \ell(P_{u_\beta,u_y})$ is a $u_\alpha u_\beta$-walk of odd length $r - (\ell(P_{u_\alpha,u_y}) - \ell(P_{u_y,u_\beta})) \leq r < n + r - 4$.

Case 3. *The vertices u_α and u_β lie on different double-cycle.* Suppose u_α lies on the cycle with base $\{u_{2i}, u_{2i+1}\}$ and u_β lies on the cycle with base $\{u_{2j}, u_{2j+1}\}$ with $i < j$. Let P_{u_α,u_β} be the $u_\alpha u_\beta$-path of length $d(u_\alpha, u_\beta)$. We note that the vertices u_{2i+1} and u_{2j} must lie on the path P_{u_α,u_β}.

If $\ell(P_{u_\alpha,u_{2i+1}}) \leq \ell(P_{u_{2j},u_\beta})$, then $\ell(P_{u_\alpha,u_{2i+1}}) + \ell(P'_{u_{2j},u_\beta}) \leq r$. Therefore, the walk W_{u_α,u_β} that consists of the paths $P_{u_\alpha,u_{2i+1}}$, $P_{u_{2i+1},u_{2j}}$, and P'_{u_{2j},u_β} is an odd $u_\alpha u_\beta$-walk of length $\ell(W_{u_\alpha,u_\beta}) \leq n + r - 5$.

If $\ell(P_{u_\alpha,u_{2i+1}}) \geq \ell(P_{u_{2j},u_\beta})$, then $\ell(P'_{u_\alpha,u_{2i+1}}) + \ell(P_{u_{2j},u_\beta}) \leq r$. Therefore, the walk W_{u_α,u_β} that consists of the paths $P'_{u_\alpha,u_{2i+1}}$, $P_{u_{2i+1},u_{2j}}$, and P_{u_{2j},u_β} is an odd $u_\alpha u_\beta$-walk of length $\ell(W_{u_\alpha,u_\beta}) \leq n + r - 5$.

Case 4. *The vertex $u_\alpha = u_1$ and u_β lies on a cycle C_r or the vertex u_α lies on a cycle C_r and $u_\beta = u_n$.* We first consider the case where $u_\alpha = u_1$ and u_β lies on some cycle C_r. Let the base of the cycle C_r be the edge $\{u_{2i}, u_{2i+1}\}$ for some i. Then the walk W_{u_α,u_β} that consist of the paths $P_{u_1,u_{2i}}$ and the path P'_{u_{2i},u_β} is an odd $u_\alpha u_\beta$-walk of length $\ell(W_{u_\alpha,u_\beta}) \leq n + r - 4$. If u_α lies on some cycle C_r with base $\{u_{2j}, u_{2j+1}\}$, the the W_{u_α,u_β} that consists of the paths $P'_{u_\alpha,u_{2j+1}}$ and P_{u_{2j+1},u_β} is an odd $u_\alpha u_\beta$-walk of length $\ell(W_{u_\alpha,u_\beta}) \leq n + r - 4$.

Therefore, from Case 1-4 we conclude that for each pair of vertices u_α and u_β there is an odd $u_\alpha u_\beta$-walk of length $\ell(W_{u_\alpha,u_\beta}) \leq n + r - 4$. Since W_{u_α,u_β} is of odd length, it can be extended to a $u_\alpha u_\beta$-walk of length exactly $n + r - 4$. Therefore, $\exp(DA(C_{r,n}) \leq n + r - 4$. □

We now discuss the scrambling index of the double alternate circular snake graphs.

Theorem 4. *Let $n \geq 4$ is an even positive integer and $r \geq 3$ is an odd positive integer. Then $k(DA(C_{r,n})) = (n + r - 3)/2$.*

Proof. Since $d(u_1, u_n) = n-1$ is odd, the shortest even u_1u_n-walk $W_{u_1u_n}$ is the walk that consists of the paths $P_{v_1,v_{2i}}$, $P'_{u_{2i},u_{2i+1}}$ and P_{u_{2i+1},u_n} of length $\ell(W_{u_1,u_n}) = n - 2 + r - 1 = n + r - 3$. Hence $k_{u_1,u_n}(DA(C_{r,n})) = (n + r - 3)/2$. This implies $k(DA(C_{r,n})) \geq (n + r - 3)/2$.

We now show that $k(DA(C_{r,n})) \leq (n+r-3)/2$. Considering Proposition 2 it suffices to show that for any pair of vertices u_α and u_β in $DA(C_{r,n})$ there is a $u_\alpha u_\beta$-walk of length $n + r - 3$. We have shown in the proof of Theorem 3 that for each pair of vertices u_i and u_j there is a u_iu_j-walk of length exactly $n + r - 4$. Let u_α and u_β be any two vertices in $DA(C_{r,n})$ and let u_γ be a vertex in $DA(C_{r,n})$ such that $\{u_\gamma, u_\beta\}$ is an edge of $DA(C_{r,n})$. By Theorem 3 there is a $u_\alpha u_\gamma$-walk of length $n + r - 4$. Since $\{u_\gamma, u_\beta\}$ is an edge, there is a $u_\alpha u_\beta$-walk of length exactly $n + r - 3$. Thus $k(DA(C_{r,n})) \leq (n + r - 3)/2$. □

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