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Some structural properties of vector valued φ-function sequence space

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Abstract. The sequence space $W(M)$, where M is an *Orlicz function* was introduced by Parashar and Choudhary [1] and Maddox [2]. Let f be *φ-function* and X be a Banach space. In this work, we introduce vector valued sequence space defined by f, denoted by $W(X, f)$. We study some topological properties and inclusion relations of this space.

1. Introduction and Preliminaries

An *Orlicz function* is a continuous, convex, non-decreasing function defined from $[0, \infty)$ to itself such that $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [3] introduced the sequence space $\ell^{\exists}(M)$ using *Orlicz function* M as follows

$$\ell^{\exists}(M) = \left\{ x = (x_k) : x_k \in \mathbb{R}\ \forall \mathrm{k} \in \mathbb{N} \text{ and } \exists \rho > 0 \text{ such that } \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

The space $\ell^{\exists}(M)$ equipped with the *Luxemburg norm*

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

becomes a Banach space which is called an *Orlicz sequence space*. The space $\ell^{\exists}(M)$ is closely related to the space ℓ_p with $1 \le p < \infty$,

$$\ell_p = \left\{ x = (x_k) : x_k \in \mathbb{R}\ \forall \mathrm{k} \in \mathbb{N} \ \text{ and } \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}$$

which is an *Orlicz sequence space* with $M(x) = x^p$. In the mathematical literature there exists various modifications of these definitions, where ℓ is replaced by another solid sequence space (see [4–6]). A sequence space X is said to be solid (or normal) if $(\lambda_k x_k) \in X$, whenever $(x_k) \in X$ and for all sequences (λ_k) of scalars with $|\lambda_k| \le 1$ for all $k \in \mathbb{N}$.

A norm $\|\cdot\|$ on a normal sequence space X is said to be *absolutely monotone norm* if $x = (x_k), y = (y_k) \in X$ and $|x_k| \le |y_k|$ for all $k \in \mathbb{N}$ implies $\|x_k\| \le \|y_k\|$. The norm

$$\|x\|_\infty = \sup |x_k|$$

over the classical sequence space $\ell\infty$, c, c_0 and the norm

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$$

over ℓ_∞ for $p \geq 1$ are absolutely monotone.

A completed normed space X is said to be a *BK-space* if the function $p_k : X \to \mathbb{R}$ where $p_k(x) = x_k$ is continuous in X for every $x = (x_k) \in X$ and every $k \in \mathbb{N}$. An *AK-space* X with the norm $\|\cdot\|$ is a *BK-space* and $\|x - x^{[n]}\| \to 0$ as $n \to \infty$ for every $x \in X$, where $x^{[n]}$ denotes the $n-th$ section of x.

Let X be a vector space. The collection of all vector valued sequences denoted by $\Omega(X)$. Any vector subspace of $\Omega(X)$ is called vector valued sequence space. The studies on vector valued sequence spaces are done by Rath and Srivastava [7], Das and Choudhary [8], Leonard [9], Srivastava and Srivastava [10] and many others.

A function $f : \mathbb{R} \to [0, \infty)$ which is continuous, vanishing at zero, non-decreasing on $[0, \infty)$ and even is called $\varphi - function$. A $\varphi - function$ f is said to satisfy $\Delta_2 - condition$ (written as $f \in \triangle_2$ for shortly), if there exists $K > 0$ such that $f(2x) \leq K f(x)$ for every $x \geq 0$.

A functional $\rho : X \to [0, \infty)$ is called a convex modular if $\rho(x) = 0 \Leftrightarrow x = 0$, even, $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. In this case, we say that X is a convex modular space.

Let $X = (X, \|\cdot\|_X)$ be a *Banach space* with an absolutely monotone norm $\|\cdot\|_X$ and f is a $\varphi - function$. Using convex $\varphi - function$ f, we introduce the following set, denoted by $W(X, f)$.

$$W(X, f) = \left\{ x = (x(i)) \in \Omega(X) : \rho_f\left(\frac{x(i) - \ell}{\alpha}\right) \to 0 \text{ as } i \to \infty, \text{ for some } \alpha > 0 \text{ and } \ell \in X \right\}$$

where

$$\rho_f(x(i)) = \frac{1}{m}\sum_{i=1}^{m} f(\|x(i)\|_X), \text{ for every } x(i) \in X$$

is a *convex modular.*

A function $g : X \to \mathbb{R}$ is said to be *paranorm* if $g(\theta) = 0$, $g(x) \geq 0$, $g(x+y) \leq g(x)+g(y)$, even and every scalar sequence (λ_n) with $|\lambda_n - \lambda| \to 0$ and every sequence (x_n) with $g(x_n - x) \to 0$ implies $g(\lambda_n x_n - \lambda x) \to 0$ for all $\lambda \in \mathbb{R}$ and $x \in X$, where θ is the zero in the linear space X. The notion of *paranormed sequence space* was introduced by Nakano [11] and Simons [12]. Later on it was further investigated by Rath and Tripathy [13], Tripathy and Sen [14].

In this work, we investigate some of topological properties of the set $W(X, f)$ equipped with a paranorm that we will define and study some inclusion relations of this set.

2. Main Results

In this section we examine some topological properties and inclusion relations of the set $W(X, f)$.

Lemma 1. *If $x, y \in X$ such that $0 \leq x \leq y$, then $\rho_f(x) \leq \rho_f(y)$.*

Theorem 2.1. *If $\varphi - function$ f satisfies the $\Delta_2 - condition$ and convex, then $W(X, f)$ is a linear space.*

Proof. Let $x = (x(i))$, $y = (y(i)) \in W(X, f)$, then there exist $\alpha_1, \alpha_2 > 0$ and $\ell_1, \ell_2 \in X$ such that

$$\rho_f\left(\frac{x(i) - \ell_1}{\alpha_1}\right) \to 0 \text{ and } \rho_f\left(\frac{y(i) - \ell_2}{\alpha_2}\right) \to 0 \text{ as } i \to \infty$$

Take $\alpha = \max\{\alpha_1, \alpha_2\}$ and $\ell = \ell_1 + \ell_2$. Considering Lemma 1 and the property of $\varphi - function$, we have

$$\rho_f\left(\frac{(x(i)+y(i))-\ell}{\alpha}\right) \le \rho_f\left(\left|\frac{x(i)-\ell_1}{\alpha_1}\right| + \left|\frac{y(i)-\ell_2}{\alpha_2}\right|\right)$$

Since ρ_f is convex and $f \in \Delta_2$, there exists $\frac{K_1}{2}, \frac{K_2}{2} > 0$ such that

$$\rho_f\left(\frac{(x(i)+y(i))-\ell}{\alpha}\right) \le \frac{K_1}{2}\rho_f\left(\frac{x(i)-\ell_1}{\alpha_1}\right) + \frac{K_2}{2}\rho_f\left(\frac{y(i)-\ell_2}{\alpha_2}\right)$$

Consequently, $\rho_f\left(\frac{(x(i)+y(i))-\ell}{\alpha}\right) \to 0$ as $i \to \infty$. Hence, $x + y \in W(X, f)$. Let $\beta \in \mathbb{R}$ and $x = (x(i)) \in W(X, f)$, then there exists $\alpha > 0$ and $\ell \in X$ such that

$$\rho_f\left(\frac{x(i)-\ell}{\alpha}\right) \to 0 \text{ as } i \to \infty$$

Choose $p = \beta\ell$. For $\beta = 0$, is clear that $\rho_f\left(\frac{\beta x(i)-p}{\alpha}\right) \to 0$ as $i \to \infty$. Now, assume that $\beta \neq 0$. Since $f \in \Delta_2$, then by using *Archimedian* there exists $n_0 \in \mathbb{N}$ and $K > 0$ such that

$$\rho_f\left(\frac{\beta x(i)-p}{\alpha}\right) \le K^{n_0}\rho_f\left(\frac{x(i)-\ell}{\alpha}\right) \to 0 \text{ as i} \to \infty$$

Therefore, $\beta x \in W(X, f)$ and the proof is complete. □

Theorem 2.2. *A function* $g : W(X, f) \to \mathbb{R}$ *with*

$$g(x) = \inf\left\{\alpha > 0 : \rho_f\left(\frac{x(i)}{\alpha}\right) \le 1\right\}$$

is a paranorm.

Proof. It is easy to show that $g(\theta) = 0$, $g(x) \ge 0$ and $g(-x) = g(x)$, for every $x \in W(X, f)$, where θ is the zero in the linear space $W(X, f)$. We shall now show the subaddivity of g. Let $x = (x(i))$, $y = (y(i)) \in W(X, f)$, then there exist $\alpha_1, \alpha_2 > 0$ such that

$$\rho_f\left(\frac{x(i)}{\alpha_1}\right) \le 1 \text{ and } \rho_f\left(\frac{y(i)}{\alpha_2}\right) \le 1$$

Take $\alpha = \max\{2\alpha_1, 2\alpha_2\}$. Considering Lemma 1 and using the convexity of ρ_f, we have

$$\begin{aligned}\rho_f\left(\frac{x(i)+y(i)}{\alpha}\right) &\le \frac{1}{2}\rho_f\left(\frac{x(i)}{\alpha_1}\right) + \frac{1}{2}\rho_f\left(\frac{y(i)}{\alpha_2}\right)\\ &\le \rho_f\left(\frac{x(i)}{\alpha_1}\right) + \rho_f\left(\frac{y(i)}{\alpha_2}\right)\end{aligned}$$

Therefore, $g(x + y) \le g(x) + g(y)$ for every $x, y \in W(X, f)$. Finally, we show that scalar multiplication is continuous. Let (λ_n) be any scalar sequence and $(x_n(i)) \subset W(X, f)$, with $|\lambda_n - \lambda| \to 0$ and $g(x_n(i) - x(i)) \to 0$ as n $\to \infty$. Considering Lemma 1 and using the the convexity of ρ_f, we have

$$\begin{aligned}\rho_f\left(\frac{\lambda_n x_n(i) - \lambda x(i)}{\alpha}\right) &\le \rho_f\left(\left|\frac{(\lambda_n-\lambda)x_n(i)}{\alpha}\right| + \left|\frac{\lambda(x_n(i)-x(i)}{\alpha}\right|\right)\\ &\le \frac{1}{2}\rho_f\left(2|\lambda_n-\lambda|\left|\frac{x_n(i)}{\alpha}\right|\right) + \frac{1}{2}\rho_f\left(2|\lambda|\left|\frac{x_n(i)-x(i)}{\alpha}\right|\right)\\ &\le \rho_f\left(2|\lambda_n-\lambda|\left|\frac{x_n(i)}{\alpha}\right|\right) + \rho_f\left(2|\lambda|\left|\frac{x_n(i)-x(i)}{\alpha}\right|\right)\end{aligned}$$

Therefore,

$$\begin{aligned}
g(\lambda_n x_n(i) - \lambda x(i)) &= \inf\left\{\alpha > 0 : \rho_f\left(\frac{\lambda_n x_n(i) - \lambda x(i)}{\alpha}\right) \le 1\right\} \\
&\le 2|\lambda_n - \lambda| \inf\left\{\alpha^* = \left(\frac{\alpha}{2|\lambda_n - \lambda|}\right) > 0 : \rho_f\left(\frac{x_n(i)}{\alpha^*}\right) \le 1\right\} \\
&\quad + 2|\lambda| \inf\left\{\alpha^{**} = \left(\frac{\alpha}{2|\lambda|}\right) > 0 : \rho_f\left(\frac{x_n(i) - x(i)}{\alpha^{**}}\right) \le 1\right\} \\
&= 2|\lambda_n - \lambda|\ g(x_n(i)) + 2|\lambda|\ g(x_n(i) - x(i)) \to 0
\end{aligned}$$

Hence, $g(\lambda_n x_n(i) - \lambda x(i)) \to 0$. This completes the proof of the theorem. □

Theorem 2.3. *The linear space $W(X, f)$ is a complete paranormed sequence space.*

Proof. Let (x_n) be any *Cauchy* sequence in $W(X, f)$ where $(x_n) = (x_n(i)) = (x_n(1), x_n(2), ...)$. This implies for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $m \ge n \ge n_0$, we get $g(x_m - x_n) < \epsilon$. Consequently, $\rho_f\left(\frac{x_m(i)-x_n(i)}{\epsilon}\right) \le 1$. Since ρ_f is convex, we have $\rho_f(x_m(i) - x_n(i)) \le \epsilon$.

Using the continuity of f, it follows that $\|x_m(i) - x_n(i)\|_X < \epsilon$ for every $\epsilon > 0$. Hence, for every fixed i, the sequence $(x_n(i))$ is a *Cauchy* sequence in X. It converges since X is complete. Say, $x_n(i) \to x(i)$ as $n \to \infty$. Using these limits, we define $x = (x(i))$ and show that $x \in W(X, f)$ and $g(x_n - x) \to 0$. Since $X = (X, \|\cdot\|_X)$ is a Banach space, we get

$$\|x_m(i) - x(i)\|_X = \|x_m(i) - \lim_{n\to\infty} x_n(i)\|_X = \lim_{n\to\infty} \|x_m(i) - x_n(i)\|_X < \epsilon^2$$

Since $(x_n(i)) \in W(X, f)$, there exists $\alpha > 0$ and $\ell \in X$ such that

$$\rho_f\left(\frac{x_n(i) - \ell}{\alpha}\right) \to 0 \text{ as } i \to \infty$$

Using the continuity of f, we obtain

$$\rho_f\left(\frac{x(i) - \ell}{\alpha}\right) = \rho_f\left(\frac{\lim_{n\to\infty} x_n(i) - \ell}{\alpha}\right) = \lim_{n\to\infty} \rho_f\left(\frac{x_n(i) - \ell}{\alpha}\right) \to 0 \text{ as } i \to \infty$$

It follows that $x \in W(X, f)$. We will show that $g(x_n - x) \to 0$. Since f is continuous, then

$$\rho_f\left(\frac{x_n(i) - x(i)}{\alpha}\right) = \rho_f\left(\frac{x_n(i) - \lim_{m\to\infty} x_m(i)}{\alpha}\right) \le 1$$

Therefore, $g(x_n - x) = \inf\left\{\alpha > 0 : \rho_f\left(\frac{x_n(i)-x(i)}{\alpha}\right) \le 1\right)$. Hence, there exists sequence $\left(\frac{c}{2^n}\right), n \ge 1$, for a real number c with $g(x_n - x) < \frac{c}{2^n}$, for every $n \ge 1$. Therefore, we get $g(x_n - x) \to 0$. We can conclude that $W(X, f)$ is a complete paranormed space. □

Theorem 2.4. *The linear space $W(X, f)$ is an AK space.*

Proof. Let $x = (x(i)) \in W(X, f)$, then there exists $\alpha > 0$ and $\ell \in X$ such that

$$\frac{1}{m}\sum_{i=1}^{m} f\left(\left\|\frac{x(i) - \ell}{\alpha}\right\|_X\right) \to 0 \text{ as i} \to \infty$$

It follows that for every $i = 1, ..., m$, we have $\|x(i) - \ell\|_X \to 0$, as $i \to \infty$. Consequently,

$$\frac{1}{m}\sum_{i=n}^{m} f\left(\left\|\frac{x(i+1)-\ell}{\alpha}\right\|_X\right) \to 0 \text{ as i} \to \infty$$

Hence, $\rho_f\left(\frac{x-x^{[n]}}{\alpha}\right) \to 0$ as $n \to \infty$, where $x^{[n]}$ denotes the $n-th$ section of x. Therefore, for $\epsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we obtain $\rho_f\left(\frac{x-x^{[n]}}{\alpha}\right) \leq 1$. It follows that $g(x - x^{[n]}) \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 2.5. *Let f and g be two $\varphi-functions$, then*
(i) $W(X, f) \subseteq W(X, g \circ f)$
(ii) $W(X, f) \cap W(X, g) \subseteq W(X, f + g)$

Proof. (i) Let $x = (x(i)) \in W(X, f)$, then there exists $\alpha > 0$ and $\ell \in X$ such that

$$\frac{1}{m}\sum_{i=1}^{m} f\left(\left\|\frac{x(i)-\ell}{\alpha}\right\|_X\right) \to 0 \text{ as i} \to \infty$$

Hence, for every $\epsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, we have

$$\frac{1}{m}\sum_{i=1}^{m} f\left(\left\|\frac{x(i)-\ell}{\alpha}\right\|_X\right) < \epsilon$$

It follows that for every $i = 1, ..., m$, we have $f\left(\left\|\frac{x(i)-\ell}{\alpha}\right\|_X\right) \to 0$ as i $\to \infty$. Since g is a $\varphi - function$, we have $g\left(f\left(\left\|\frac{x(i)-\ell}{\alpha}\right\|_X\right)\right) \to 0$ as i $\to \infty$. Hence, $\rho_{g\circ f}\left(\frac{x(i)-\ell}{\alpha}\right) \to 0$ as i $\to \infty$. This implies $x \in W(X, g \circ f)$. This concludes the proof.
(ii) The result of this point is obvious. □

References

[1] Parashar S and Choudhary B 1994 *Indian J. Pure and App. Math.* **25** 419-28.
[2] Maddox I J 1988 *Elements of Functional Analysis* (Cambridge: Cambridge University Press).
[3] Lindenstrauss J and Tzafriri L 1971 *Israel J. Math.* **10** 379-90.
[4] Ghosh D and Srivastava P D 1999 *Glasnik Matematicki* **34** 253-261.
[5] Malkowsky E and Savas E 2000 *Arch. Math.* **36** 219-228.
[6] Kolk E 1994 *Tartu UI Toimetised* **970** 65-72.
[7] Rath A and Srivastava P D 1996 *Ganita* **47** 1-12.
[8] Das N R and Choudhary 1992 *Bul. Calcutta Math. Soc.* **84** 47-54.
[9] Leonard I E 1976 *J. Math. Anal. App.* **54** 245-65.
[10] Srivastava J K and Srivastava B K 1996 *India J. Pure App. Math.* **27** 73-84.
[11] Nakano 1951 *Proc. Japan Acad.* vol 27 p 508-12.
[12] Simons S 1965 *Proc. London Math. Soc.* p 422-436.
[13] Rath D and Tripathy B C 1994 *Indian J. Pure App. Math.* **25** 381-86.
[14] Tripathy B C and Sen M 2001 *Indian J. Pure App. Math.* **32** 1689-94.
[15] Mahmut Isik 2011 *Tamsui Oxford J. of Information and Mathematical Sciences* **27** 137-48.
[16] Özdemir M K and Solak I 2006 *Thai J. of Math.* **27** 93-105.