

Nonlinear evolution of Benjamin-Feir wave group based on third order solution of Benjamin-Bona-Mahony equation

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Abstract. This study concerns on the evolution of trichromatic wave group . It has been known that the trichromatic wave group undergoes an instability during its propagation, which results wave deformation and amplification on the waves amplitude. The previous results on the KdV wave group showed that the nonlinear effect will deform the wave and lead to large wave whose amplitude is higher than the initial input. In this study we consider the Benjamin-Bona-Mahony equation and the theory of third order side band approximation to investigate the peaking and splitting phenomena of the wave groups which is initially in trichromatic signal. The wave amplitude amplification and the maximum position will be observed through a quantity called Maximal Temporal Amplitude (MTA) which measures the maximum amplitude of the waves over time.

1. Introduction

The evolution of nonlinear wave can be portrayed as the occurrence of spontaneous high amplitude waves. When the wave height exceeds the significant wave height of measured wave train by factor more than 2.2, the wave is known as an extreme or rogue wave [1]. The extreme wave can appear from nowhere and then disappear without a trace [2]. There are some mechanisms of extreme wave occurrence; wave-current interaction, geometrical focusing, soliton collision, focusing due to dispersion, and modulational instability [2–4]. Modulational instability happens as the effect of nonlinearity. The instability causes growing modulation of the wave's envelope due to the energy focusing. This extreme wave behavior has been actively studied and appears not only in hydrodynamic context but also in optical waves, quantum electronics, and plasma physics [5–9].

Extreme wave phenomenon may occur to the bichromatic wave group which is a superposition of two monochromatic waves with the same amplitudes and slightly different frequencies. The wave which is initially in bichromatic form experiences instability and focusing during its propagation. New side bands occur and modulate the wave causing an amplification to the



amplitude [10]. This mechanism had been also investigated numerically and experimentally in [11] and [12]. Marwan [13] studied the deformation of this wave based on the third order asymptotic series solution of Korteweg de Vries (KdV) equation. The wave propagates and undergoes amplitude amplification until it reaches the maximum peaking and then followed by the decrease at its amplitude. The similar approach up to the fifth order solution of KdV equation was also inspected in [14]. The result suggests that the higher order of the solution gives higher maximum amplification. However, these wave's maximum amplifications are still far below the ones obtained from numerical software HUBRIS which is governed based on Laplace equation [12] and Boussinesq equation [15]. The evolution of bichromatic wave group was also investigated in [16] on the basis of Benjamin-Bona-Mahony (BBM) equation. The study suggests that the initial amplitude and frequency difference of the bichromatic signal may affects the amplitude amplification factor and the maximum position.

An instability also happens to the trichromatic wave group. Trichromatic wave is formed when a monochromatic wave is perturbed by a pair of side bands with smaller amplitude and slightly different frequencies. This wave is also known as Benjamin-Feir type wave. The study of the deformation of this wave was studied in [17] by employing the KdV equation as the wave model. The amplification and extreme position was inspected by utilizing a quantity called maximal temporal amplitude (MTA). This quantity measures the maximum wave's amplitude at each spatial position during the observation time [13–15]. The evolution of rouge wave has been also investigated through the Nonlinear Schrodinger (NLS) equation. NLS also yields solutions which experience instability during the wave propagation [18–20]. This equation appears as the envelope of wave groups of many nonlinear equations such as KdV [6, 21–23], BBM [24], and KP [9].

In this study, we are interested in investigating the evolution of a wave group which is initially in the trichromatic wave form. The wave will be developed based on the third order asymptotic solution of BBM equation. The amplitude amplification and the maximum position will be inspected by generating the MTA of the wave.

2. Third Order Side Band Solution of BBM Model

Benjamin-Bona-Mahony (BBM) equation was developed as a revised model of KdV equation. Benjamin et al [25] explained that the classic KdV equation failed the unidirectional assumption when the wave number value is high. The regularized long wave model they proposed can explain the wave physical behavior better. BBM equation is mathematically stated as [25],

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0, \quad (1)$$

where η is elevation, x and t are respectively spatial and time variables.

We develop the solution of (1) through asymptotic expansion up to the third order. Suppose that the solution of (1) is expressed in the asymptotic series below,

$$\eta(x, t) = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} + O(\varepsilon^4), \quad (2)$$

where ε is real number. The solution for $\eta^{(i)}$ will be determined at each order. Furthermore, we consider trichromatic wave as the initial wave form:

$$\eta(x, t) = \sum_{p=1}^3 a_p e^{i\theta_p} + c.c. \quad (3)$$

where a_p represents amplitude, $\theta_p = \omega_p t - k_p x + \psi_p$, ω_p is frequency, k_p is wavenumber, ψ_p is phase shift, and *c.c.* is the complex conjugate. To investigate the deformation of

the wave as the peaking phenomena, we define the trichromatic wave as a monochromatic wave being perturbed by a pair of side bands with smaller amplitude. Hence we assign $a_2 > a_1$, $a_1 = a_3$, and $\omega_1 = \omega_2 - \nu$, $\omega_3 = \omega_2 + \nu$, where ν is small real number. Also, to resolve the secular terms which may occur at higher order, we applied Linstedt-Poincare method and expand the wave number in the series $k_p = k_p^{(0)} + \varepsilon k_p^{(1)} + \varepsilon^2 k_p^{(2)} + O(\varepsilon^3)$, for each $p = 1, 2, 3$.

We substitute the wave number series and (2) into (1), collect the terms at the same order of ε , and solve the equation separately at every order. For the first order $O(\varepsilon)$, we obtain an equation equivalent to the linear form of equation (1). Assuming the solution is in the trichromatic wave form (3) yields the dispersion relation $\Omega(k_p^{(0)}) = \omega_p = k_p^{(0)} / \left(1 + (k_p^{(0)})^2\right)$, for $p = 1, 2, 3$.

On the second order $O(\varepsilon^2)$, there appears secular terms which will resonance with the homogeneous solution of first order equation and cause the solution of higher order have higher amplitude. To eliminate these terms, we set $k_p^{(1)} = 0$. Furthermore, on the equation of $O(\varepsilon^3)$, the value of $k_p^{(2)}$ needs to be assigned to treat the resonant terms that are appeared. The value of the correction wave number reads

$$k_p^{(2)} = \frac{-(K_1 + K_2)}{a_p - 2a_p k_p^{(0)} \omega_p},$$

where $K_1 = a_p k_p^{(0)} \left(a_1^2 (A_{+,1p} + 2A_{-,11}) + a_2^2 (A_{+,2p} + 2A_{+,22}) + a_3^2 (A_{+,3p} + 2A_{-,33}) + a_t^2 (A_{+,pt} + A_{-,pt} + A_{-,tp}) + a_s^2 (A_{+,ps} + A_{-,ps} + A_{-,sp}) \right)$,

$$K_2 = \begin{cases} a_t^2 a_s (A_{+,tt} + A_{-,ts} + A_{-,st}) (2k_t^{(0)} - k_s^{(0)}), & p = 1 \\ a_p a_t a_s ((A_{+,st} + A_{+,ts}) + (A_{-,sp} + A_{-,ps}) + (A_{-,tp} + A_{-,pt})) (k_t^{(0)} + k_s^{(0)} - k_p^{(0)}), & p = 2 \\ a_s^2 a_t (A_{+,ss} + A_{-,st} + A_{-,ts}) (2k_s^{(0)} - k_t^{(0)}), & p = 3 \end{cases}$$

$$t = (p \bmod 3) + 1, s = (p - 2 \bmod 3) + 1.$$

We will investigate the evolution of the trichromatic wave modulated by the side bands. Hence, assuming $a_2 = a$, $a_1 = a_3 = \delta a$, δ is small real number. Equation (3) as the solution of first order equation can be rewritten as

$$\eta^{(1)} = 2(\delta a \cos \theta_1 + a \cos \theta_2 + \delta a \cos \theta_3) = 2a(1 + 2\delta \cos \Delta\theta) \cos \theta_2, \quad (4)$$

where $\Delta\theta = (\theta_3 - \theta_1)/2 = (\nu t - \kappa t + \Delta\psi)$. We obtain that the solution of $O(\varepsilon^2)$ is waves consisting of frequencies which are far from the frequencies of trichromatic waves (4). These waves may not significantly affect the wave modulation and the extreme position. The waves with higher frequency also comes up in third order solution. Omitting these terms, we only consider the waves with near frequencies as the side bands. Hence, the third order side band waves and the free waves are stated as follow,

$$\begin{aligned} \eta_{sb}^{(3)} &= \delta^2 a^3 \left((B_{-,123} + B_{-,213}) \cos(\theta_1 - \Delta\theta_r) + B_{-,112} \cos(\theta_1 - \Delta\theta_l) + \right. \\ &\quad \left. (B_{-,1231} + B_{-,321}) \cos(\theta_3 + \Delta\theta_l) + B_{-,332} \cos(\theta_3 + \Delta\theta_r) \right) + \\ &\quad \delta^3 a^3 \left(B_{-,113} \cos(\theta_1 - 2\Delta\theta) + B_{-,331} \cos(\theta_3 + 2\Delta\theta) \right), \\ \eta_{sb,fw}^{(3)} &= \delta^2 a^3 \left((B_{-,123} + B_{-,213}) \cos(\varphi(\theta_1 - \Delta\theta_r)) + B_{-,112} \cos(\varphi(\theta_1 - \Delta\theta_l)) + \right. \\ &\quad \left. (B_{-,1231} + B_{-,321}) \cos(\varphi(\theta_3 + \Delta\theta_l)) + B_{-,332} \cos(\varphi(\theta_3 + \Delta\theta_r)) \right) + \\ &\quad \delta^3 a^3 \left(B_{-,113} \cos(\varphi(\theta_1 - 2\Delta\theta)) + B_{-,331} \cos(\varphi(\theta_3 + 2\Delta\theta)) \right), \end{aligned}$$

where $\Delta\theta_l = \theta_2 - \theta_1$, $\Delta\theta_r = \theta_3 - \theta_2$, $\varphi(\theta_p) = \omega_p t - \Omega^{-1}(\omega_p)x + \psi_p$,

$$B_{+,pqr} = \frac{C_+ A_{+,pq}}{D_+}, \quad B_{-,pqr} = \frac{C_- (A_{+,pq} + A_{+,pr} + A_{+,rq})}{D_-}, \quad C_{\pm} = \left(k_p^{(0)} + k_q^{(0)} \pm k_r^{(0)} \right),$$

$$D_{\pm} = (\omega_p + \omega_q \pm \omega_r) - (k_p^{(0)} + k_q^{(0)} \pm k_r^{(0)}) + (k_p^{(0)} + k_q^{(0)} \pm k_r^{(0)})^2 (\omega_p + \omega_q \pm \omega_r),$$

$$A_{\pm,pq} = \pm k_q^{(0)} / \left((\omega_p \pm \omega_q) - (k_p^{(0)} \pm k_q^{(0)}) + (k_p^{(0)} \pm k_q^{(0)})^2 (\omega_p \pm \omega_q) \right), \text{ and } A_{-,pq} = 0 \text{ for } p = q.$$

Finally, the side band solution is $\eta(x, t) \approx \eta^{(1)} + \eta_{sb}^{(3)} - \eta_{sb, fw}^{(3)}$.

3. Extreme Position and Amplitude Amplification

The extreme position of the side band modulated waves will be inspected by employing Maximal Temporal Amplitude (MTA). MTA can be mathematically stated as [13].

$$\text{MTA}(x) = \max_t \eta(x, t)$$

The result will be presented in the laboratory measure. Thus, nondimensional BBM equation stated in (1) needs to be converted to its physical variables (see [16]). Here, we consider a water tank with water depth 5 m and gravitational acceleration 9.8 m/s^2 . Taking $a = 0.2 \text{ m}$ and $\delta = 0.5 \text{ m}$ gives the initial amplitude of the trichromatic signal at 0.8 m , while setting $\omega_{2,lab} = 0.5 \text{ rad/s}$, $\nu = 0.02 \text{ rad/s}$ yields wave numbers $k_{1,lab}^{(0)} = 7.00 \text{ rad/m}$, $k_{2,lab}^{(0)} = 6.71 \text{ rad/m}$, $k_{3,lab}^{(0)} = 6.44 \text{ rad/m}$. We ignore the contribution of the phase shift, thus $\psi_1 = \psi_2 = \psi_3 = 0$.

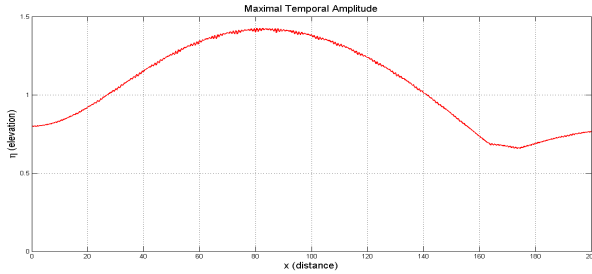


Figure 1. MTA of trichromatic wave signal for $a_{lab} = 0.2 \text{ m}$, $\delta = 0.5$, $\omega_{2,lab} = 0.5 \text{ rad/s}$, $\nu = 0.02 \text{ rad/s}$.

Figure 1 exhibits the maximum amplitude of the signal along the spatial axis. The wave is initially at the amplitude 0.8 m at $x = 0$ comes through an amplitude increase until it reaches the maximum position at $x = 81.5 \text{ m}$ with maximum amplitude $\eta_{max} = 1.42 \text{ m}$. Thus, the wave is amplified as much as 1.78 times its initial amplitude. The signal deformation at some positions is presented in figure 2. Figure 2(a) is the trichromatic signal at $x = 0$. New groups of waves appear at $x = 40 \text{ m}$ (figure 2(b)), modulate the wave and cause the peaking on the amplitude. The signal at maximum position is shown in figure 2(c). After reaching the maximum peaking, the wave amplitude starts to decrease as presented in figure 2(d). The peaking phenomenon is affected by the appearance of new side band waves during the propagation of the wave. The spectra of the waves forming the propagating wave are illustrated in figure 3. It can be observed that at the initial position $x = 0$, there is trichromatic wave form and during the propagation there grows new side bands modulating the trichromatic signal.

Some parameters affect the maximum position and amplitude amplification factor (AAF). AAF is calculated by comparing the highest value of MTA with the value of MTA at the initial position [14]. Table 1 presents the values of maximum elevation and position of the wave for some values of the monochromatic wave's frequencies $\omega_{2,lab}$ and the frequency differences ν . The results indicate that higher frequency yields higher amplitude amplification while extending

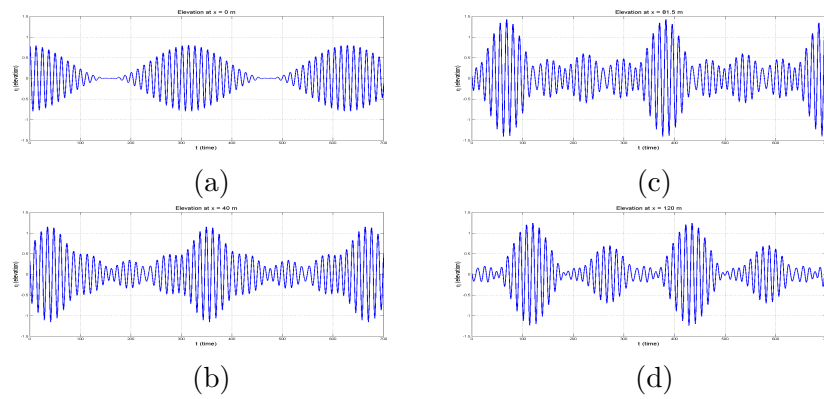


Figure 2. Wave signals on time domain at (a) $x = 0$ m (b) $x = 40$ m (c) $x = 81.5$ m (d) $x = 120$ m for parameter values $a_{lab} = 0.2$ m, $\delta = 0.5$, $\omega_{2,lab} = 0.5$ rad/s, $\nu = 0.02$ rad/s.

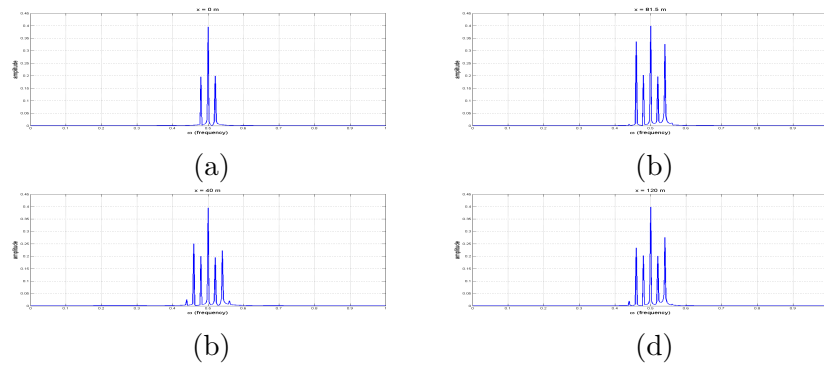


Figure 3. Amplitude spectra of the wave signals at (a) $x = 0$ m (b) $x = 40$ m (c) $x = 81.5$ m (d) $x = 120$ m for parameter values $a_{lab} = 0.2$ m, $\delta = 0.5$, $\omega_{2,lab} = 0.5$ rad/s, $\nu = 0.02$ rad/s.

the distance of the maximum peaking from the initial position. On the contrary, the higher frequency difference will decrease the amplification but makes the maximum peaking position closer. Meanwhile, the change in amplitude a will also influence the amplification and maximum position. While the higher initial amplitude gives higher amplification, it shortens the distance of wave traveling to reach the maximum peaking position. These results is presented in table 2.

Table 1. Maximum position and AAF for some values of frequencies and frequency differences with $a_{lab} = 0.2$ m, $\delta = 0.5$.

$\omega_{2,lab}$ (rad/s)	ν (rad/s)	x_{max} (m)	η_{max} (m)	AAF
0.4	0.02	55.5	1.25	1.56
0.4	0.03	31	1.00	1.24
0.45	0.02	69.5	1.34	1.68
0.45	0.03	42	1.06	1.33
0.5	0.02	81.5	1.42	1.78
0.5	0.03	57	1.13	1.41

Table 2. Maximum position and AAF for some values of amplitudes a_{lab} with $\delta = 0.5, \omega_{2,lab} = 0.5 \text{ rad/s}, \nu = 0.02 \text{ rad/s}$.

$a_{lab}(\text{m})$	$x_{max}(\text{m})$	$\eta_{max}(\text{m})$	AAF
0.16	108.5	1.00	1.57
0.18	94.5	1.21	1.68
0.23	66.5	1.79	1.94
0.25	59	2.05	2.05

4. Conclusion

We have derived a model of nonlinear wave propagation which is initially in trichromatic wave form based on BBM model. The wave deforms during its propagation and experiences a peaking and splitting phenomena. The amplitude amplification can be observed through the MTA. The higher wave frequency for the initial signal will produce higher AAF but farther maximum peaking position. Conversely, the closer frequency difference among the supporting wave of the trichromatic signal decreases the amplitude amplification but makes the maximum position nearer to the initial position. Meanwhile, the higher initial amplitude gives higher AAF and closer maximum MTA.

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