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Modification of 2-D Time-Domain Shallow Water Wave Equation using Asymptotic Expansion Method

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Abstract. Generally, research on the tsunami wave propagation model can be conducted by using a linear model of shallow water theory, where a non-linear side on high order is ignored. In line with research on the investigation of the tsunami waves, the Boussinesq equation model underwent a change aimed to obtain an improved quality of the dispersion relation and non-linearity by increasing the order to be higher. To solve non-linear sides at high order is used a asymptotic expansion method. This method can be used to solve non linear partial differential equations. In the present work, we found that this method needs much computational time and memory with the increase of the number of elements.

1. Introduction

Wave propagation, like tsunami waves, can be caused by undersea movement. The problem of tsunami waves that caused coastal damage has been widely studied by experts. The damage caused by the waves relates to how far, high and rapid the spreading of water when entering the coastal area. Studies conducted to understand the evolution of tsunami wave propagation have been done by using shallow water theories, Boussinesq equations, non-hydrostatic wave equations, and Navier-Stokes equations [1, 2]. The use of the Boussinesq equation for tsunami wave propagation has also been used by Groesen and Klopman [3] and Adytia [4]. The Boussinesq equation is an equation that modeled wave propagation on surfaces propagating in two opposite directions [5].

One form of the Boussinesq equation, especially for shallow waters known as the shallow water wave equation [6–8]. In line with the study of tsunami wave investigation, the Boussinesq equation model undergoes a change aimed at improving the quality of dispersion relation and inclination by increasing the order to be higher. The complexity in the numerical implementation arising from the application of the high order Boussinesq model would be modified and solved by asymptotic expansion methods.

2. Theory

Tsunami waves have an initial wavelength of several tens or hundreds of km. In governing equations, nonlinear convective tribes and ocean floor friction are relatively small and negligible,

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while the influence of frequency dispersion depends on wavelength and can be included in linear model governing equations. Non linear effect is not significant, while the effect of frequency dispersion can be important for shorter waves. Therefore, deep ocean tsunami propagation can be modeled by linear Boussinesq equations. Furthermore, tsunamis move across the ocean from the initial location of a tsunami to a coastal area where the water depth is relatively shallower.

A number of governing equations are used to simulate the propagation of tsunami waves, such as linear Boussinesq equations. After ignoring the nonlinear terms, the linear Boussinesq equation can be written as follows [9–11]:

$$\partial_t u + g\partial_x \eta + u\partial_x u - \frac{H^2}{3}\partial_{xxt} u = 0 \tag{1}$$

$$\partial_t \eta + \partial_x (Hu) + \partial_x(\eta u) = 0 \tag{2}$$

3. Methods

In this article, asymptotic expansion method was used to approximate a analytic solution of shallow water wave equation. To apply this method, the first step had to do is to make an assumption related to the form of asymptotic method that will be used. Here, the asymptotic expansion forms of elevation $\eta(x,t)$ and velocity $u(x,t)$ are written as the following

$$\begin{aligned} \eta(x,t) &\approx \varepsilon\eta^{(1)}(x,t) + \varepsilon^2\eta^{(2)}(x,t) + \varepsilon^3\eta^{(3)}(x,t) \\ u(x,t) &\approx \varepsilon u^{(1)}(x,t) + \varepsilon^2 u^{(2)}(x,t) + \varepsilon^3 u^{(3)}(x,t) \end{aligned} \tag{3}$$

where ε is a small parameter in the first order that states the ratio between wave amplitude and depth. η^j and u^j respectively represent solutions for elevation and velocity at the $j - order$, $j = 1, 2, \ldots$ [5].

4. Results and Discussion

The form of asymptotic expansion of equation(3) is substituted into equations (1) dan (2). After each side is grouped according to the order of each ε and assigns the value of each part to zero, it is obtained

$$O(\varepsilon): \quad \partial_t u^{(1)} + g\partial_x\eta^{(1)} - \frac{H^2}{3}\partial_{xxt}u^{(1)} = 0 \tag{4}$$

$$\partial_t\eta^{(1)} + H\partial_x u^{(1)} = 0 \tag{5}$$

$$O(\varepsilon^2): \quad \partial_t u^{(2)} + g\partial_x\eta^{(2)} - \frac{H^2}{3}\partial_{xxt}u^{(2)} = -u^{(1)}\partial_x u^{(1)} \tag{6}$$

$$\partial_t\eta^{(2)} + H\partial_x u^{(2)} = -(\eta^{(1)}\partial_x u^{(1)} + u^{(1)}\partial_x\eta^{(1)}) \tag{7}$$

$$O(\varepsilon^3): \quad \partial_t u^{(3)} + g\partial_x\eta^{(3)} - \frac{H^2}{3}\partial_{xxt}u^{(3)} = -u^{(1)}\partial_x u^{(2)} - u^{(2)}\partial_x u^{(1)} \tag{8}$$

$$\partial_t\eta^{(3)} + H\partial_x u^{(3)} = -\partial_x(\eta^{(1)}u^{(2)} + \eta^{(2)}u^{(1)}) \tag{9}$$

4.1. First-order solution

Eliminating $u^{(1)}$ from equations (4 and 5), the following equation can be obtained in term of η.

$$\partial_{tt}\eta^{(1)} - gH\partial_{xx}\eta^{(1)} = \frac{gH^3}{3}\partial_{xxxx}\eta^{(1)} \tag{10}$$

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Solution for first-order equations (10) can be written in the form of cosine equations as follows

$$\eta^{(1)}(x,t) = A_1 \cos(kx - \omega t) \tag{11}$$

where A_1, k, and ω, respectively, are wave amplitude, the wave number, and dispersion relation of wave. Hereafter, the form of this first-order solution is derived respectively to x and t, so that it is obtained

$$\frac{\omega^2}{k^2} = gH\left(1 + \frac{(kH)^2}{3}\right) \tag{12}$$

4.2. Second-order solution
In the second order we obtain the non homogeneous linear differential equation as follows

$$\partial_{tt}\eta^{(2)} - gH\partial_{xx}\eta^{(2)} = \frac{gH^3}{3}\partial_{xxxx}\eta^{(2)} - \frac{H}{2}\partial_{xx}(u^{(1)})^2 - \partial_{xt}(u^{(1)}\eta^{(1)})) \tag{13}$$

The following two solutions are taken as solutions to the equation (13).

$$\begin{aligned} \eta^{(2)}(x,t) &= A_2 \cos(2(kx - \omega t)) \\ u^{(1)}(x,t) &= B_1 \cos(kx - \omega t) \end{aligned} \tag{14}$$

Equation 14 is derived respectively to x and t so we get

$$\begin{aligned} \partial_{xx}(u^{(1)})^2 &= -2B_1^2k^2\cos(2(kx - \omega t)) \\ \partial_{xt}(u^{(1)}\eta^{(1)}) &= 2A_1B_1k\omega\cos(2(kx - \omega t)) \end{aligned} \tag{15}$$

Then equations (12, 14 and 15) are substituted to equation (13), so that the value of A_2 can be written as the following

$$A_2 = \frac{3}{5}\left(\frac{HB_1^2k^2 + 2A_1B_1k\omega}{gH^3k^4}\right) \tag{16}$$

4.3. Third-order solution
As the second-order, in the third order we obtain the non homogeneous linear differential equation as follows

$$\partial_{tt}\eta^{(3)} - gH\partial_{xx}\eta^{(3)} = \frac{gH^3}{3}\partial_{xxxx}\eta^{(3)} + H\partial_{xx}(u^{(1)}u^{(2)}) - \partial_{xt}(\eta^{(1)}u^{(2)} + \eta^{(2)}u^{(1)})) \tag{17}$$

By applying the same working principle as in the previous order, for the third order equation, it is known that the multiplication of the inhomogeneous parts yields

$$\begin{aligned} \partial_{xx}(u^{(1)}u^{(2)}) &= -B_1B_2k^2\left(9\cos(3(kx-\omega t))+\tfrac{1}{9}\cos(kx-\omega t)\right) \\ \partial_{xt}(\eta^{(1)}u^{(2)}+\eta^{(2)}u^{(1)}) &= (A_1B_2+A_2B_1)k\omega\left(9\cos(3(kx-\omega t))+\tfrac{1}{9}\cos(kx-\omega t)\right) \end{aligned} \tag{18}$$

where $u^{(2)}(x,t) = B_2\cos(kx-\omega t)$. Therefore, the solution form for equation (17) can be written as

$$\eta^{(3)}(x,t) = C_1\cos(3(kx-\omega t)) \tag{19}$$

The substitution of equations (18 and 19) into equation (17) gives the following results

$$C_1 = \frac{1}{60}\left(\frac{B_1B_2k+(A_1B_2+A_2B_1)\omega}{H^2k^3\cos(3(kx-\omega t))}\right)\left(9\cos(3(kx-\omega t))+\tfrac{1}{9}\cos(kx-\omega t)\right) \tag{20}$$

Based on the results obtained as written in equation (20), then equation (19) can be rewritten to

$$\eta^{(3)}(x,t) = \frac{1}{60}\left(\frac{B_1B_2k+(A_1B_2+A_2B_1)\omega}{H^2k^3}\right)\left(9\cos(3(kx-\omega t))+\tfrac{1}{9}\cos(kx-\omega t)\right) \tag{21}$$

4.4. First solution

Three solutions for each order have been completed. Thus, the Boussinesq equation solution for tsunami wave propagation obtained by asymptotic expansion method until the third order can be rewritten as

$$\begin{aligned} \eta(x,t) = \quad & \varepsilon A_1\cos(kx-\omega t) \\ & +\varepsilon^2A_2\cos(2(kx-\omega t)) \\ & +\varepsilon^3C_1\left(9\cos(3(kx-\omega t))+\tfrac{1}{9}\cos(kx-\omega t)\right) \end{aligned} \tag{22}$$

with

$$C_1 = \frac{1}{60}\left(\frac{B_1B_2k+(A_1B_2+A_2B_1)\omega}{H^2k^3}\right) \tag{23}$$

and

$$B_2 = \frac{A_2\omega-\frac{1}{2}A_1B_1k}{kH} \tag{24}$$

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4.5. Correction of Wave Numbers

Regarding the essence of applying the asymptotic expansion method is to approach a value, which is an analytical solution of the shallow water equation. That is, the approximate solutions must be convergent. Therefore, to obtain a convergent solution, further correction of other parameters of the wave equation is needed, in this case, that is the wave numbers (k). To do this, the procedure applied is the same as of for obtaining the solution of each order, as described in the previous section.

Assuming that the wave numbers (k) has the following expansion shape,

$$k \approx k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)} \tag{25}$$

Now, equation (25) is substituted to equation (22) to replace value of (k), so that equation (22) is rewritten as the following

$$\begin{aligned} \eta(x,t) = \quad & \varepsilon A_1 \cos\Big((k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)})x - \omega t\Big) \\ & + \varepsilon^2 A_2 \cos\Big(2((k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)}) - \omega t)\Big) \\ & + \varepsilon^3 C_1 \Big(9\cos(3((k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)})x - \omega t)) \\ & + \tfrac{1}{9}\cos((k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)})x - \omega t)\Big) \end{aligned} \tag{26}$$

Approximate solutions (26) with their each derivative of the variables x and t are resubstituted to equations (10), (13), and (17) so that we get these following results.

- First-order, $O(\varepsilon)$:
  The result in first order is a new shape of dispersion relation, namely

$$\frac{\omega^2}{(k^{(0)})^2} = gH\left(1 + \frac{(k^{(0)}H)^2}{3}\left(1 + 2k^{(0)}\right)\right) \tag{27}$$

- Second-order, $O(\varepsilon^2)$:
  In this order, it was obtained $k^{(1)} = 0$, so that the value of coefficient A_2 in the equation (14) can be rewritten as below

$$A_2 = \frac{k^{(2)}}{2}\left(\frac{HB_1^2 k^{(0)} + \omega A_1 B_1}{gH(k^{(0)})^2 - \frac{4}{3}gH^3(k^{(0)})^4 - \omega^2}\right) \tag{28}$$

- Third-order, $O(\varepsilon^3)$:
  The value of $k^{(1)} = 0$ obtained in the second-order causes some terms of the third-order, in which each order is still dependent on the value $k^{(1)}$, can be eliminated. Therefore, the new expansion form of k can be rewritten as follows.

$$k = k^{(0)} + \varepsilon^2 k^{(2)} \tag{29}$$

  where

$$k^{(2)} = -\frac{C_1}{9}\left(\frac{(k^{(0)})^2 + \omega^2 + \frac{1}{3}gH^3(k^{(0)})^4)}{A_1 k^{(0)}\,(gH(k^{(0)})^2) - 2)}\right) \tag{30}$$

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Based on all the above results, now a solution of shallow water equations solved using the asymptotic expansion method can be described as the new following form

$$\begin{aligned}\eta(x,t) = \quad & \varepsilon A_1 \cos\left((k^{(0)} + \varepsilon^2 k^{(2)})x - \omega t\right) \\ & + \varepsilon^2 A_2 \cos\left(2((k^{(0)} + \varepsilon^2 k^{(2)}) - \omega t)\right) \\ & + \varepsilon^3 C_1 \left(9\cos(3((k^{(0)} + \varepsilon^2 k^{(2)})x - \omega t)) \right. \\ & \left. + \tfrac{1}{9}\cos((k^{(0)} + \varepsilon^2 k^{(2)})x - \omega t)\right) \qquad (31)\end{aligned}$$

The coefficient of $k^{(2)}$ has an important role in the expansion of the wave number k. It also can be interpreted as a consequence of the use of the asymptotic expansion method to obtain the approximate solution of shallow water equations. In other words, the convergence of the solution will be achieved when the wave number k is corrected by $k^{(2)}$.

5. Summary

Based on the modified result, asymptotic expansion method is very appropriate in approaching Boussinesq analytical solution of high order equation. However, the solution that has been obtained can not be said as a valid solution. This is because of the resonance part, which is the shape that looks like the form of the first-order solution. Therefore, in order for the solution to be valid, further correction is required on other parameters in the Boussinesq wave equation.

Meanwhile, the result of modifying the Boussinesq equation into a form that can be solved by asymptotic expansion method shows that many new coefficients and variables appear. Of course, the emergence of these new coefficients and variables enlarge the calculation time and also increase the amount of computer memory used for computer simulation.

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