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On some new normed sequence spaces

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Abstract. The sequence spaces $(c_0)_\Lambda$, c_Λ, and $(\ell_\infty)_\Lambda$ was introduced and studied by Mursaleen and Noman [11]. In the present paper, for M is a generalization of Orlicz function, we extend the spaces Mursaleen and Noman's to $[c_0(M)]_\Lambda$, $[c(M)]_\Lambda$, and $[\ell_\infty(M)]_\Lambda$, respectively, and investigate some topological properties of these spaces. Finally, we determine the necessary and sufficient conditions of an infinite matrix A belonging to classes $\big(c_0(M), c_0(M)\big)$, $\big(c(M), c(M)\big)$, and $\big(\ell_\infty(M), \ell_\infty(M)\big)$.

1. Introduction and Preliminaries

By ω, we denote the space of all sequences of real or complex numbers. Any linear subspace of ω is called a *sequence space*. We shall write c_0, c, and ℓ_∞ for the spaces of all convergent to zero, convergent, and bounded sequences, respectively.

A sequence space X is called a *BK space* provided X is a complete normed space and a function $p_k : X \to \mathbb{R}$ defined by $x \mapsto p_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$ (see [5]).

The sequence spaces c_0, c, and ℓ_∞ are BK spaces equipped with sup-norm $\|\cdot\|_\infty$ given by

$$\|x\| = \sup_{k\in\mathbb{N}} |x_k|.$$

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and X, Y be the sequence spaces. The map A from X into Y is said to be *matrix transformation* if $Ax = (A_n(x))$ exists in Y where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \text{ converges for all } n \in \mathbb{N} \text{ and all } x \in X. \tag{1}$$

We denote (X, Y) as the class of all infinite matrices that map X into Y. Thus, $A \in (X, Y)$ if and only if (1) hold, and $Ax \in Y$ for all $x \in X$. The theory of matrix transformation deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y.

For a sequence space X, the *matrix domain* of an infinite matrix A in X is a sequence space defined by

$$X_A = \Big\{x = (x_k) \in \omega : Ax \in X\Big\}.$$

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been studied by several authors, e.g., Altay and Başar [1], Mursaleen et al. [2], Mursaleen and Noman [11, 12], Malkowsky [9], Malkowsky and Savaş [10]. Mursaleen and Noman [11] introduced Λ-matrix and constructed the matrix domains on Λ-matrix in the classical sequence spaces c_0, c, and ℓ_∞. They examined some topological properties, established inclusion relations concerning with those spaces, determined their α-, β-, γ-duals, and characterized some relate matrix classes.

On the other side, Lindenstrauss and Tzafriri [7] introduced the sequence space defined by Orlicz function as follows :

$$\ell_M = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) \ \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called *Orlicz sequence space*. The space ℓ_M equipped with norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space.

Using the matrix domain X_Λ defined by Mursaleen and Noman [11], in this work, we introduce Λ-matrix domain for the sequences generated by a generalization of Orlicz function M, denoted by $[X(M)]_\Lambda$ where $X \in \{c_0, c, \ell_\infty\}$. Furthermore, we investigate some topological properties of these spaces over the norm spaces, and give the necessary and sufficient conditions on an infinite matrix A belonging to classes $\big(c_0(M), c_0(M)\big)$, $\big(c(M), c(M)\big)$, and $\big(\ell_\infty(M), \ell_\infty(M)\big)$.

2. Results

2.1. The Sequence Space $[X(M)]_\Lambda$

A function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *generalization of Orlicz function* which is vanishing at zero, non decreasing, and continuous. A generalization of Orlicz function M is said to satisfy Δ_2-*condition* for all values of x if there exists a constant $K > 0$ such that $M(2x) \leq KM(x)$ for $x \geq 0$. Furthermore, in [11], Mursaleen and Noman defined the infinite matrix $\Lambda = (\lambda_{nk})$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n} & ; 0 \leq k \leq n \\ 0 & ; k > n \end{cases} \tag{2}$$

where $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive reals tending to infinity, that is, $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$. By using (2), in the present section we define the sequence space $[X(M)]_\Lambda$ where $X \in \{c_0, c, \ell_\infty\}$ and M is a generalization of Orlicz function, and prove that these sequence spaces according to its norm are complete normed spaces. These sequence spaces are as follows:

$$[c_0(M)]_\Lambda = \left\{ x = (x_k) \in \omega : (\exists \ \rho > 0) \ M\left(\frac{|\Lambda_n(x)|}{\rho}\right) \to 0, n \to \infty \right\},$$

$$[c(M)]_\Lambda = \left\{ x = (x_k) \in \omega : (\exists \ \rho > 0, \ l \in \mathbb{R}) \ M\left(\frac{|\Lambda_n(x)|}{\rho}\right) \to l, n \to \infty \right\}, \text{ and}$$

$$[\ell_\infty(M)]_\Lambda = \left\{ x = (x_k) \in \omega : (\exists \ \rho > 0) \ \sup_{n \in \mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho}\right) < \infty \right\}.$$

Now, we may begin with the following results which is essential in the text.

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2.2. Linear Topological Structure of $[X(M)]_\Lambda$

In this section, we examine some topological properties of the sequence spaces defined above.

Theorem 2.1. *If M is a convex function, then the sequence space $[X(M)]_\Lambda$ for $X \in \{c_0, c, \ell_\infty\}$ is linear space over the set of real numbers $\mathbb{R}$.*

Proof. We prove the theorem for $X = \ell_\infty$. Let $x, y \in [\ell_\infty(M)]_\Lambda$ and $\alpha, \beta \in \mathbb{R}$, then there exist some positive ρ_1 and ρ_2 such that

$$\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho_1}\right) < \infty \text{ and } \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(y)|}{\rho_2}\right) < \infty.$$

Take $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$, then for a convex function M we get

$$\begin{aligned}
\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(\alpha x + \beta y)|}{\rho}\right) &\le \sup_{n\in\mathbb{N}} M\left(\frac{|\alpha\Lambda_n(x)|}{\rho} + \frac{|\beta\Lambda_n(y)|}{\rho}\right) \\
&\le \sup_{n\in\mathbb{N}} M\left(\frac{|\alpha||\Lambda_n(x)|}{2|\alpha|\rho_1} + \frac{|\beta||\Lambda_n(y)|}{2|\beta|\rho_2}\right) \\
&\le \frac{1}{2}\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho_1}\right) + \frac{1}{2}\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(y)|}{\rho_2}\right) < \infty.
\end{aligned}$$

This proves that $[\ell_\infty(M)]_\Lambda$ is linear space. □

It is easy to show that $[c_0(M)]_\Lambda$ and $[c(M)]_\Lambda$ are also linear spaces.

Theorem 2.2. *If M satisfy Δ_2-condition, then the space $[X(M)]_\Lambda$ for $X \in \{c_0, c, \ell_\infty\}$ is complete normed space equipped with the norm defined by*

$$\|x\|_{[X(M)]_\Lambda} = \inf\left\{\rho > 0 : \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho}\right) \le 1\right\} \tag{3}$$

Proof. We prove the theorem for $X = \ell_\infty$ and the other cases will follow similarly. Let $x, y \in [\ell_\infty(M)]_\Lambda$. It is easily seen that $\|x\|_{[\ell_\infty(M)]_\Lambda} \ge 0$. Next, if $x = 0$, then obviously $\|x\|_{[\ell_\infty(M)]_\Lambda} = 0$. Conversely, suppose $\|x\|_{[\ell_\infty(M)]_\Lambda} = 0$, then for every $\epsilon > 0$ we get $\|x\|_{[\ell_\infty(M)]_\Lambda} < \epsilon$. This implies there exists some ρ_0 with $0 < \rho_0 < \epsilon$ such that

$$\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\epsilon}\right) < \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho_0}\right) \le 1.$$

Since M is a generalization of Orlicz function, it follows that for every $\epsilon > 0$ and for every $n \in \mathbb{N}$,

$$|\Lambda_n(x)| = \left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k\right| = 0.$$

Under the assumption that $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers, it is easy to check by mathematical induction that $x_k = 0$ for every $k \in \mathbb{N}$. Thus, $x = 0$.

Furthermore, let $x \in [\ell_\infty(M)]_\Lambda$ and $\alpha \in \mathbb{R}$. If $\alpha = 0$, it is clear that the homogeneous property of the norm holds. Assume $\alpha \ne 0$, we get

$$\|\alpha x\|_{[\ell_\infty(M)]_\Lambda} = |\alpha| \inf\left\{\frac{\rho}{|\alpha|} > 0 : \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\frac{\rho}{|\alpha|}}\right) \le 1\right\}.$$

This gives $\|\alpha x\|_{[\ell_\infty(M)]_\Lambda} = |\alpha|\|x\|_{[\ell_\infty(M)]_\Lambda}$.

Now, let $x, y \in [\ell_\infty(M)]_\Lambda$, then there exists some $\rho_1, \rho_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho_1}\right) \leq 1 \text{ and } \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(y)|}{\rho_2}\right) \leq 1.$$

Hence, if we choose $\rho = \rho_1 + \rho_2$, then by the properties of M, we have $M\left(\frac{|\Lambda_n(x+y)|}{\rho}\right) \leq M\left(\frac{|\Lambda_n(x)|}{\rho_1}\right) + M\left(\frac{|\Lambda_n(y)|}{\rho_2}\right)$ for all $n \in \mathbb{N}$. Consequently, $\|x+y\|_{[\ell_\infty(M)]_\Lambda} \leq \|x\|_{[\ell_\infty(M)]_\Lambda} + \|y\|_{[\ell_\infty(M)]_\Lambda}$. Hence, $[\ell_\infty(M)]_\Lambda$ is a normed space.

Now, suppose that (x^i) be any Cauchy sequence in $[\ell_\infty(M)]_\Lambda$. Then, for each $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ and ρ_0 where $0 < \rho_0 < \epsilon$ such that

$$\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x^j - x^i)|}{\epsilon}\right) < \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x^j - x^i)|}{\rho_0}\right) \leq 1.$$

Hence, for every $\epsilon > 0$, $M\big(|\Lambda_n(x^j - x^i)|\big) < \epsilon$. Consequently, $|x^j_k - x^i_k| < \epsilon$ for each $\epsilon > 0$, every $j \geq i \geq i_0$, and every $k \in \mathbb{N}$, where $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers. We see that (x^j_k) is Cauchy sequences of real numbers. Since $\mathbb{R}$ is complete, there exists $x_k \in \mathbb{R}$ such that $x^j_k \to x_k$ as $j \to \infty$ for all $k \in \mathbb{N}$. Using these limits, we define $x = (x_k)$ and show that $x \in [\ell_\infty(M)]_\Lambda$ and $x^i \to x$ as $i \to \infty$ in $[\ell_\infty(M)]_\Lambda$. From (3), we have for all $i \geq i_0$

$$\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x^i - x)|}{\rho_0}\right) = \lim_{j\to\infty} \sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x^i - x^j)|}{\rho_0}\right) \leq 1.$$

We obtain $\|x^i - x\| < \epsilon$ for every $i \geq i_0$. This shows that

$$x^i \to x \text{ as } i \to \infty \text{ in } [\ell_\infty(M)]_\Lambda.$$

Since $x^i \in [\ell_\infty(M)]_\Lambda$ and M satisfy Δ_2-condition, we have

$$\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x)|}{\rho}\right) \leq \frac{K_1}{2}\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x - x^i)|}{\rho}\right) + \frac{K_2}{2}\sup_{n\in\mathbb{N}} M\left(\frac{|\Lambda_n(x^i)|}{\rho}\right) < \infty$$

for some $K_1, K_2 > 0$. It is show that $x \in [\ell_\infty(M)]_\Lambda$.

Since (x^i) was an arbitrary Cauchy sequence in $[\ell_\infty(M)]_\Lambda$, this proves completeness of $[\ell_\infty(M)]_\Lambda$. □

Now, the following result is immediate by Theorem 2.2.

Theorem 2.3. *$[X(M)]_\Lambda$ is a BK space where $X \in \{c_0, c, \ell_\infty\}$ and $[X(M)]_\Lambda$ is a AK space where $X = c_0$.*

2.3. The Certain Classes of Matrix Transformations $\big(X(M), X(M)\big)$

Theorem 2.4. *$A \in \big(c(M), c(M)\big)$ if and only if for $\rho > 0$ the following conditions are held :*

(i) $\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$,

(ii) $\lim_{n\to\infty} M\left(\frac{|a_{nk}|}{\rho}\right) = \alpha_k$ *exists for each* $k \in \mathbb{N}$, *and*

(iii) $\lim_{n\to\infty} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}|}{\rho}\right) = \alpha$ *exist.*

Proof. For proving the necessity, suppose that $A \in \big(c(M), c(M)\big)$. If we choose $x = (x_k)$ by $x_k = sgn(a_{nk})$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$, then $\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) = \sup_{n\in\mathbb{N}} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right) < \infty$. This shows that the condition *(i)* holds. Furthermore, if we take $x = (x_j)$ where for all $j \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$x_j = e_j^{[k]} = \begin{cases} 1 & ; j = k \\ 0 & ; j \neq k \end{cases}$$

Since $Ax \in c(M)$ for every $x \in c(M)$, there exists $\rho > 0$ such that $\lim_{n\to\infty} M\left(\frac{|\sum_{j=0}^{\infty} a_{nj}x_j|}{\rho}\right)$ exist. Hence, for all $k \in \mathbb{N}$, $\lim_{n\to\infty} M\left(\frac{|\sum_{j=0}^{\infty} a_{nj}x_j|}{\rho}\right) = \lim_{n\to\infty} M\left(\frac{|a_{nk}|}{\rho}\right)$. It is shown that $\lim_{n\to\infty} M\left(\frac{|a_{nk}|}{\rho}\right)$ exists for each $k \in \mathbb{N}$, and the condition *(ii)* holds. Next, since $x = (x_k) = (1, 1, 1, \cdots)$ belongs to $c(M)$, the condition *(iii)* holds.

For sufficiency, let $x_k \to r$ as $k \to \infty$ and let the conditions *(i)*, *(ii)*, and *(iii)* hold. We write

$$\sum_{k=0}^{\infty} a_{nk}x_k = \sum_{k=0}^{\infty} a_{nk}(x_k - r) + r\sum_{k=0}^{\infty} a_{nk} \text{ for all } n \in \mathbb{N}.$$

Since M satisfy Δ_2-condition, then for some $\rho > 0$ we get

$$M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right) \leq \frac{K_0}{2} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}(x_k - r)|}{\rho}\right) + \frac{K_1^{m_0+1}}{2} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}|}{\rho}\right)$$

for all $n \in \mathbb{N}$ and for some $K_0, K_1 > 0$. By *(iii)*, we get $\lim_{n\to\infty} \frac{K_1^{m_0+1}}{2} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}|}{\rho}\right) = \frac{\alpha K_1^{m_0+1}}{2}$. Further, since $x_k \to r$ as $k \to \infty$, we get

$$\frac{K_0}{2} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}(x_k - r)|}{\rho}\right) + \frac{K_1^{m_0+1}}{2} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}|}{\rho}\right) \to \alpha_1 \text{ as } n \to \infty$$

where $\alpha_1 = \frac{\alpha K_1^{m_0+1}}{2}$. It is shown that $M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right) \to \alpha_1$ as $n \to \infty$. Thus, $Ax \in c(M)$. Since for each $x \in c(M)$ implies $Ax \in c(M)$, we conclude that $A \in \big(c(M), c(M)\big)$, which proves the theorem. □

Theorem 2.5. *$A \in \big(c_0(M), c_0(M)\big)$ if and only if for $\rho > 0$ the following conditions are held :*
(i) $\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$, *and*
(ii) $\lim_{n\to\infty} M\left(\frac{|a_{nk}|}{\rho}\right) = 0$ *for each* $k \in \mathbb{N}$.

Proof. We first derive the necessary conditions *(i)* and *(ii)*. Since $x = (x_j) = (e_j^{[k]})$ belongs to $c_0(M)$, then

$$\lim_{n\to\infty} M\left(\frac{|a_{nk}|}{\rho}\right) = \lim_{n\to\infty} M\left(\frac{|\sum_{j=0}^{\infty} a_{nj}x_j|}{\rho}\right) = 0 \text{ for each } k \in \mathbb{N}.$$

It is shown that *(ii)* holds. Further, we define $x = (x_k)$ by $x_k = sgn(a_{nk})$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Thus, for some $\rho > 0$ we have

$$\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) = \sup_{n\in\mathbb{N}} M\left(\frac{|\sum_{k=0}^{\infty}|a_{nk}||}{\rho}\right) = \sup_{n\in\mathbb{N}} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right).$$

Since $\lim_{n\to\infty} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right)$ exists, then sequence $\left(M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right)\right)$ is bounded. Thus, $\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$, which yields *(i)* holds.

For proving the sufficiency, let us take any $x \in c_0(M)$. Since M is continuous, then $x_k \to 0$ as $k \to \infty$. Thus, $\lim_{n\to\infty} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right) = 0$. It is shown that $Ax \in c_0(M)$. Since for each $x \in c(M)$ implies $Ax \in c(M)$, then the infinite matrix A belongs to the class $\big(c_0(M), c_0(M)\big)$, which completes the proof. □

Theorem 2.6. $A \in \big(\ell_\infty(M), \ell_\infty(M)\big)$ *if and only if for* $\rho > 0$

$$\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty.$$

Proof. For proving the necessity, suppose that $A \in \big(\ell_\infty(M), \ell_\infty(M)\big)$, that is, for each $x \in \ell_\infty(M)$ implies $Ax \in \ell_\infty(M)$. Thus, $\sup_{n\in\mathbb{N}} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right) < \infty$. Then, define $x = (x_k)$ by $x_k = sgn(a_{nk})$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Thus, for some $\rho > 0$ and for every $n \in \mathbb{N}$, we have

$$M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) = M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right).$$

Since $\sup_{n\in\mathbb{N}} M\left(\frac{|\sum_{k=0}^{\infty} a_{nk}x_k|}{\rho}\right) < \infty$, then $\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$.

For sufficiency, suppose $\sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$ and take any $x \in \ell_\infty(M)$. Since M is non decreasing, there exists $N_1 > 0$ such that $|x_k| \le N_1 = \rho N_0$ for all $k \in \mathbb{N}$. Hence, by using Hölder inequality [8], we get

$$\sup_{n\in\mathbb{N}} M\left(\frac{|A_n(x)|}{\rho}\right) \le \sup_{n\in\mathbb{N}} M\left(\frac{\sup_{k\in\mathbb{N}}|x_k|\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right).$$

Further, by using the Archimedean property, since $N_1 \in \mathbb{R}$, then there exists $m_0 \in \mathbb{N}$ such that $N_1 \le 2^{m_0}$. Since M satisfy Δ_2-condition, we have

$$\sup_{n\in\mathbb{N}} M\left(\frac{|A_n(x)|}{\rho}\right) \le K^{m_0} \sup_{n\in\mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$$

for some $K > 0$. It is show that $Ax \in \ell_\infty(M)$. Since for each $x \in \ell_\infty(M)$ implies $Ax \in \ell_\infty(M)$, then the infinite matrix $A \in \big(\ell_\infty(M), \ell_\infty(M)\big)$, and the proof is complete. □

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