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An analytical comparison of optimal output transition and polynomial trajectories

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Abstract. This paper compares the solution of the optimal output transition approach with polynomial trajectories. Although the optimal output transition approach can guarantee the optimal output trajectory, the complex problem formulation tends to be required, especially for nonlinear systems. The aim of this paper is to investigate differences between the optimal output transition approach and the polynomial-based approach. Based on the standard first and second order linear system examples, the output transition trajectory can be obtained by using the polynomial-based approach without any significant differences when compared to the optimal output transition approach.

1. Introduction

Changing the output of the system y from the initial value y_I to the final value y_F during transition time T_{tt} is a fundamental control problem [1] and is of particular interest. Applications of the output transition problem can be found, for example, in nanopositioning using a piezo-based flexible structure system [2] and in a vertical take-off and landing (VTOL) aircraft [3]. One of the output transition approaches is the optimal output transition approach where the acceleration of the output trajectory is minimized [4]. However, obtaining the optimal output transtion trajectory normally requires a complex problem formulation, especially for nonlinear systems [5, 6]. The objective of this paper is to observe differences, if any, between the solution of the optimal output transition approach [4] and polynomial trajectories [7, 8]. Examples of the standard first and second order linear systems are provided to show the exact solution of the optimal output transition approach. According to the optimal solution of the example systems, the output trajectory can be obtained by using the polynomial-based approach without any significant differences when compared to the optimal output transition approach. This work extends the previous work that studied simulation results by using the flexible structure example in [9] to analytical results.

2. Output Transition Problem

Consider the invertible transfer function G of the linear system

$$G(s) = \frac{y(s)}{u(s)}, \tag{1}$$

where y is the output and u is the input of the system. The output transition problem is to find the output transition input u_{tran} that can be used to transition the output y from the initial value y_I to the final value y_F during transition time T_{tt}, as shown in figure 1.

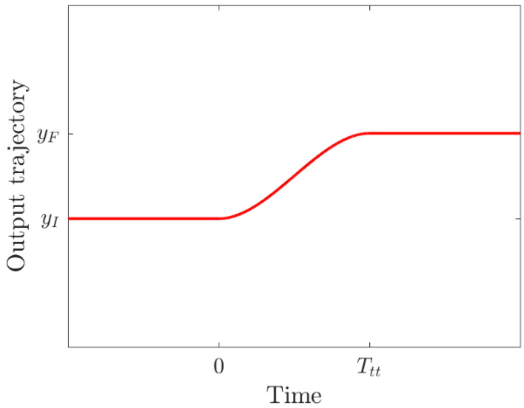

Figure 1. The output of the system y is transitioned from the initial value $y(t)=y_I, \forall t \le 0$ to the final value $y(t)=y_F, \forall t \ge T_{tt}$.

2.1. *Optimal Output Trajectory*

The optimal output transition approach as given in [4] is summarized in this section. Let the associated system dynamics be given by

$$\dot{\tilde{x}}(t)=\frac{d}{dt}\left(\tilde{x}(t)\right)=\tilde{A}\tilde{x}(t)+\tilde{B}\tilde{u}(t) \tag{2}$$

during transition time $t\in\left[0,T_{tt}\right]$. The associated state $\tilde{x}$ is given by

$$\tilde{x}(t)=\begin{bmatrix} y_{tran}(t) & \frac{d}{dt}\left(y_{tran}(t)\right) & \frac{d^2}{dt^2}\left(y_{tran}(t)\right) & \cdots & \frac{d^r}{dt^r}\left(y_{tran}(t)\right) & \frac{d^{(r+1)}}{dt^{(r+1)}}\left(y_{tran}(t)\right)\end{bmatrix}' \tag{3}$$

where $'$ represents the transpose of the matrix, r is the relative degree of the system, y_{tran} is the desired output transition trajectory with the boundary conditions

$$\tilde{x}(0)=\begin{bmatrix} y_{tran}(0) \\ \dot{y}_{tran}(0) \\ \ddot{y}_{tran}(0) \\ \vdots \\ y_{tran}^{(r+1)}(0)\end{bmatrix}=\begin{bmatrix} y_I \\ 0 \\ 0 \\ \vdots \\ 0\end{bmatrix} \text{ and } \tilde{x}(T_{tt})=\begin{bmatrix} y_{tran}(T_{tt}) \\ \dot{y}_{tran}(T_{tt}) \\ \ddot{y}_{tran}(T_{tt}) \\ \vdots \\ y_{tran}^{(r+1)}(T_{tt})\end{bmatrix}=\begin{bmatrix} y_F \\ 0 \\ 0 \\ \vdots \\ 0\end{bmatrix}, \tag{4}$$

and matrices $\tilde{A}$ and $\tilde{B}$ are given by

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(r+1)\times(r+1)} \quad \text{and } \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}_{(r\times 1)} . \tag{5}$$

Optimal output trajectory can be obtained by minimizing the cost functional J

$$\min_{\tilde{u}} \left[J_{T_{tt}} = \int_0^{T_{tt}} \left[\tilde{u}(\tau) \right]^2 d\tau \right], \tag{6}$$

where the optimal input $\tilde{u}$ can specifically be chosen as the $r+1$ time derivative of the output $y^{(r+1)}$ and as a function of transition time T_{tt}

$$\tilde{u}(t) = y_{T_{tt}}^{(r+1)}(t) = \frac{d^{(r+1)}}{dt^{(r+1)}}\left(y_{T_{tt}}(t)\right) = \tilde{B}'\left[e^{\tilde{A}'(T_{tt}-t)}\right]\left[G_{tran}^{-1}(T_{tt})\right]\left\{\tilde{x}(T_{tt}) - e^{\tilde{A}T_{tt}}\tilde{x}(0)\right\}. \tag{7}$$

The invertible controllability Gramian G_{tran}, as a function of transition time T_{tt}, is given by

$$G_{tran}(T_{tt}) = \int_0^{T_{tt}} e^{\tilde{A}(T_{tt}-\tau)}\tilde{B}\tilde{B}'e^{\tilde{A}'(T_{tt}-\tau)}d\tau . \tag{8}$$

Finally, the desired output transition trajectory y_{tran} can be found by integrating the solution of equation (7) $r+1$ times with initial conditions at time $t=0$ as given in equation (4). It is noted that equation (7) is slightly different from the filtered version of the r^{th} time derivative of the output $y_{T_{tt}}^{(r)}(t) = d^r y_{T_{tt}}(t)/dt^r$ that is used in the previous work [4].

2.2. Polynomial Trajectories

The polynomial-based approach was originally developed in [7] and is summarized in this section. Polynomial trajectories can be obtained by assuming the polynomial of the form

$$y_{tran}(t) = a_0 + a_1\left(\frac{t}{T_{tt}}\right) + a_1\left(\frac{t}{T_{tt}}\right)^2 + \ldots + a_n\left(\frac{t}{T_{tt}}\right)^n, t \in \left[0, T_{tt}\right], \tag{9}$$

where coefficients a 's can be obtained by matching the output y_{tran} and its time derivatives with the boundary conditions given in equation (4) and $n = 2r+1$ for the system with relative degree r. Alternatively, polynomial trajectories can be written as

$$y_{tran}(t) = y_I + \left(y_F - y_I\right)\sum_{i=r+1}^{2r+1} a_i \left(\frac{t}{T_{tt}}\right)^i, t \in \left[0, T_{tt}\right] \tag{10}$$

where coefficients a_i 's are given by

$$a_i = \frac{(-1)^{i-r-1}(2r+1)!}{i \times r!(i-r-1)!(2r+1-i)!} \tag{11}$$

as studied in [8]. It is noted that the output trajectory, obtained by using equation (7), is expected to be equivalent to the $r+1$ time differentiable polynomial trajectories, given in equation (10).

2.3. Output Transition Input

Substituting the output transition trajectory y_{tran} obtained by using the optimal output transition approach from equations (2) to (8) or using the polynomial-based approach in equation (10) into the system in equation (1), the output transition input u_{tran} becomes

$$u_{tran}(s) = G^{-1}(s) y_{tran}(s) \tag{12}$$

with the use of the inverse system G^{-1}.

3. First and Second Order System Examples

The standard first and second order linear systems are used to compare the solution of the optimal control approach with polynomial trajectories.

3.1. First Order System Example.

One of the systems that has relative degree $r=1$ is the first order system

$$G_1(s) = \frac{y(s)}{u(s)} = \frac{b}{s+b}, \tag{13}$$

where b is constant. The output transition input u_{tran} for the first order system in equation (13) can be given by

$$u_{tran}(t) = \frac{1}{b}\dot{y}_{tran}(t) + y_{tran}(t), t \in [0, T_{tt}], \tag{14}$$

where the output transition trajectory y_{tran} can be obtained by using the optimal output trajectory or polynomial trajectories.

3.1.1. Optimal Output Trajectory. Since the relative degree $r=1$, the matrices $\tilde{A}$ and $\tilde{B}$ in equation (5) for the first order system becomes

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{15}$$

By substituting equation (15) into equation (8), the controllability Gramian G_{tran} can be found as

$$G_{tran}(T_{tt}) = \int_0^{T_{tt}} \begin{bmatrix} (T_{tt}-\tau)^2 & T_{tt}-\tau \\ T_{tt}-\tau & 1 \end{bmatrix} d\tau = \begin{bmatrix} T_{tt}^3/3 & T_{tt}^2/2 \\ T_{tt}^2/2 & T_{tt} \end{bmatrix}. \tag{16}$$

For the system with relative degree $r=1$, the optimal input $\tilde{u}$ can be computed by using equation (5)

$$\tilde{u}(t) = \ddot{y}_{T_{tt}}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ T_{tt}-t & 1 \end{bmatrix} \begin{bmatrix} 12/T_{tt}^3 & -6/T_{tt}^2 \\ -6/T_{tt}^2 & 4/T_{tt} \end{bmatrix} \left\{ \begin{bmatrix} y_F \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & T_{tt} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_I \\ 0 \end{bmatrix} \right\} = (y_F - y_I)\left(\frac{6}{T_{tt}^2} - \frac{12}{T_{tt}^3}t\right) \tag{17}$$

for $t \in [0, T_{tt}]$. The optimal output trajectory y_{tran} can be found by integrating equation (17) twice with the initial conditions $\dot{y}_{tran}(0) = 0$ and $y_{tran}(0) = y_I$ as

$$y_{tran}(t) = y_I + (y_F - y_I)\left(3\left(\frac{t}{T_{tt}}\right)^2 - 2\left(\frac{t}{T_{tt}}\right)^3\right), t \in [0, T_{tt}]. \tag{18}$$

3.1.2. Polynomial Trajectories. For the system with relative degree $r = 1$, the coefficients a_i can be computed by using equation (11) as

$$a_2 = 3 \text{ and } a_3 = -2 \tag{19}$$

and the polynomial trajectory given in equation (10) with the coefficients a_i in equation (19) becomes equation (18) exactly.

3.2. Second Order System Example.
The second order system G_2 which has a general form of

$$G_2(s) = \frac{y(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{20}$$

where ω_n is the natural frequency and ζ is the damping ratio, has relative degree $r = 2$. The output transition input u_{tran} for the second order system in equation (20) can be given by

$$u_{tran}(t) = \frac{1}{\omega_n^2}\ddot{y}_{tran}(t) + \frac{2\zeta}{\omega_n}\dot{y}_{tran}(t) + y_{tran}(t), t \in [0, T_{tt}], \tag{21}$$

where the output transition trajectory y_{tran} can be obtained by using the optimal output trajectory or polynomial trajectory.

3.2.1. Optimal Output Trajectory. The matrices $\tilde{A}$ and $\tilde{B}$ in equation (5) for the second order system with relative degree $r = 2$ becomes

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{22}$$

By substituting equation (22) into equation (8), the controllability Gramian G_{tran} can be found as

$$G_{tran}(T_{tt}) = \int_0^{T_{tt}} \begin{bmatrix} (T_{tt}-\tau)^4/4 & (T_{tt}-\tau)^3/2 & (T_{tt}-\tau)^2/2 \\ (T_{tt}-\tau)^3/2 & (T_{tt}-\tau)^2 & T_{tt}-\tau \\ (T_{tt}-\tau)^2/2 & T_{tt}-\tau & 1 \end{bmatrix} d\tau = \begin{bmatrix} T_{tt}^5/20 & T_{tt}^4/8 & T_{tt}^3/6 \\ T_{tt}^4/8 & T_{tt}^3/3 & T_{tt}^2/2 \\ T_{tt}^3/6 & T_{tt}^2/2 & T_{tt} \end{bmatrix}. \tag{23}$$

For the system with relative degree $r = 2$, the optimal input $\tilde{u}$ can be computed by using equation (5)

$$\begin{aligned} \tilde{u}(t) = \ddot{y}_{T_{tt}}(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ T_{tt}-t & 1 & 0 \\ (T_{tt}-t)^2/2 & T_{tt}-t & 1 \end{bmatrix} \begin{bmatrix} 720/T_{tt}^5 & -360/T_{tt}^4 & 60/T_{tt}^3 \\ -360/T_{tt}^4 & 192/T_{tt}^3 & -36/T_{tt}^2 \\ 60/T_{tt}^3 & -36/T_{tt}^2 & 9/T_{tt} \end{bmatrix} \\ \left\{ \begin{bmatrix} y_F \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ T_{tt} & 1 & 0 \\ T_{tt}^2/2 & T_{tt} & 1 \end{bmatrix} \begin{bmatrix} y_I \\ 0 \\ 0 \end{bmatrix} \right\} = (y_F - y_I)\left(\frac{60}{T_{tt}^3} - \frac{360}{T_{tt}^4}t + \frac{360}{T_{tt}^5}t^2\right) \end{aligned} \tag{24}$$

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for $t \in [0, T_{tt}]$. The optimal output trajectory y_{tran} can be found by integrating equation (24) three times with the initial conditions $\ddot{y}_{tran}(0) = \dot{y}_{tran}(0) = 0$ and $y_{tran}(0) = y_I$ as

$$y_{tran}(t) = y_I + (y_F - y_I)\left(10\left(\frac{t}{T_{tt}}\right)^3 - 15\left(\frac{t}{T_{tt}}\right)^4 + 6\left(\frac{t}{T_{tt}}\right)^5\right), t \in [0, T_{tt}]. \quad (25)$$

3.2.2. Polynomial Trajectories. For the system with relative degree $r = 2$, the coefficients a_i can be computed by using equation (11) as

$$a_3 = 10, \; a_4 = -15, \text{ and } a_5 = 6 \quad (26)$$

and the polynomial trajectory given in equation (10) with the coefficients a_i in equation (26) is exactly similar to equation (25).

4. Conclusion and Discussion

Optimal output trajectory was compared with polynomial trajectories. Based on the standard first and second order system examples, the polynomial-based approach can be used to obtain the output transition trajectory without any significant difference when compared to the optimal output trajectory. Since this work seeks to find the exact solution of the optimal output transition approach, only the examples of the first and second order linear systems are considered. However, polynomial trajectories could be used not only for higher order linear systems, as simulated with the flexible structure example in [9], but also for nonlinear systems, as studied in [6, 8], instead of the optimal output transition approach, as proposed in [4].

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