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Existence of Nonsteady Planar Ideal Flows in Anisotropic Plasticity

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Abstract. Ideal plastic flows are those for which all material elements follow minimum work paths. The general equations for steady and nonsteady planar ideal flows in Tresca solids have been given elsewhere. The present paper focuses on nonsteady planar ideal flows in anisotropic plasticity. In particular, the existence of such flows is proven under a certain assumption concerning the orientation of principal stress trajectories at the initial instant. It is also shown that the system of kinematic equations is hyperbolic. This system can be treated separately from the stress equations. The original ideal flow theory is widely used as the basis for inverse methods for the preliminary design of metal forming processes driven by minimum plastic work. The new theory extends this area of application to anisotropic materials.

1. Introduction

The ideal flow theory has long been associated with solids satisfying Tresca's yield criterion and its associated flow rule [1]. The ideal flow condition is that trajectories of the major principal stress are fixed in the material. This condition is an additional equation to the standard system of equations of plasticity theory. However, in many cases it is possible to show that a large class of solutions exists for this over-determined system of equations. In particular, a proof of the existence of steady ideal flows in Tresca solids has been provided in [2]. This result has been extended to nonsteady flows in [1]. Ideal flows result in maximum uniformity and minimum resistance and, therefore, are useful for the preliminary design of material forming processes [3]. A review of the ideal flow theory and ideal flow solutions has been provided in [4]. Solutions for anisotropic materials are only available in the case of sheet metal forming (for example, [5]). On the other hand, plastic anisotropy is a common property of many materials in bulk material forming. It is therefore of interest and importance to extend the bulk ideal flow theory to such materials. A proof of the existence of steady planar ideal solutions for orthotropic materials has been given in [6]. This ideal flow theory is based on the model of anisotropic

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plasticity proposed in [7] and the yield criterion proposed in [8]. The objective of the present paper is to extend the proof given in [6] to nonsteady flow.

2. Material model

The plane strain yield criterion of any incompressible anisotropic material which complies with the principle of maximum plastic dissipation is expressed solely in terms of the stress variables $s=\left(\sigma_{xx}-\sigma_{yy}\right)/2$ and $\tau=\sigma_{xy}$ [8]. Here σ_{xx}, σ_{yy} and σ_{xy} are the components of the stress tensor in a Cartesian coordinate system (x, y). This yield criterion is represented as a unique contour in the Mohr stress space (Fig. 1[a]). Let us introduce another Cartesian coordinate system (x', y') in the plane of deformation. The corresponding stress components are denoted as σ'_{xx}, σ'_{yy} and τ'_{xy}. Consequently, $\tau'=\tau'_{xy}$ and $s'=\left(\sigma'_{xx}-\sigma'_{yy}\right)/2$. Let γ be the angle between the $s'-$ axis and the outward normal to the yield contour (Fig. 1[a]). It is evident that there are primed coordinate systems in which $\gamma=0$. One of such systems is shown in Fig.1[b]. Assume that the axes of the primed system are tangent to the principal stress trajectories at a given point *M*. Then, point *M* corresponds to point *P* in the Mohr stress space and, according to the associated flow rule, the principal stress and principal strain rate directions coincide at *M*. In this case the yield criterion can be written as

$$\sigma_1-\sigma_2=K \tag{1}$$

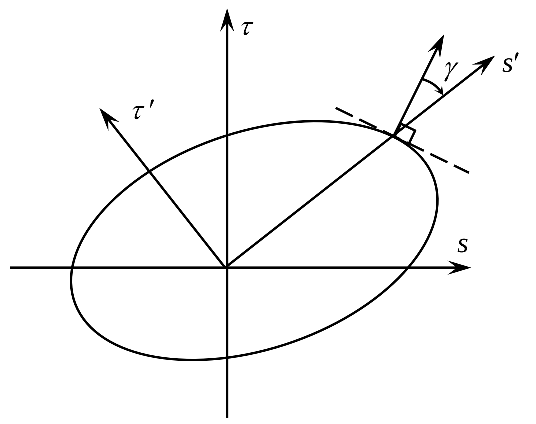
where σ_1 and σ_2 are the principal stresses and *K* is a material constant. The subsequent analysis is valid for deformation processes in which the axes of the primed system are tangent to the principal stress trajectories at each point of the plastic region at the initial instant. Therefore, the yield criterion (1) is also valid in the entire plastic region at the initial instant. The evolution of plastic anisotropy is governed by the law proposed in [7].

3. Existence of ideal flow

Let us introduce two coordinate systems; namely, a Cartesian coordinate system (x, y) and a principal lines system (ξ,η) (i.e. the coordinate curves of this coordinate system coincide with trajectories of the principal stress directions). In the principal lines coordinate system $\sigma_\xi\equiv\sigma_1$ and $\sigma_\eta\equiv\sigma_2$. Therefore, equation (1) becomes

$$\sigma_\xi-\sigma_\eta=K \tag{2}$$

at the initial instant. According to the ideal flow condition the principal lines coordinate system is Lagrangian. In this case, the evolution law for plastic anisotropy proposed in [7] shows that equation (2) is valid throughout the process of deformation. Thus, in order to demonstrate the existence of ideal flow, it is necessary to prove that the yield criterion (2) and its associated flow rule are compatible with the ideal flow condition. The associated flow rule is equivalent to two equations: (i) the incompressibility equation and (ii) zero shear strain rate in the (ξ,η) coordinate system. Let h_ξ and



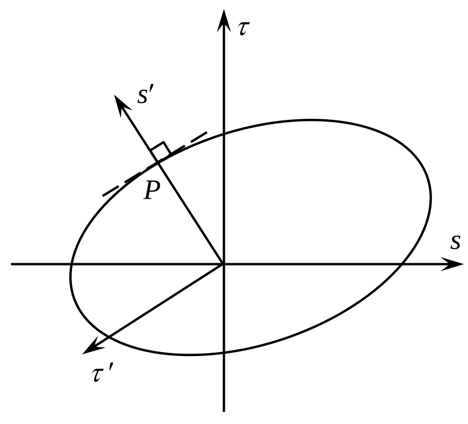


Figure 1. Yield contour in Mohr plane.

h_η be the scale factors for the coordinate curves of the (ξ,η) coordinate system. It has been shown in [6] that in the case of the yield criterion (2) it is always possible to choose the principal lines coordinate system such that

$$h_\xi h_\eta = 1. \tag{3}$$

This equation generalizes the corresponding equation in isotropic plasticity derived in [9]. Since the (ξ,η) coordinate system is Lagrangian, the incompressibility equation requires that equation (3) is satisfied throughout the process of deformation. Since the shear strain rate vanishes in the (ξ,η) coordinate system, the $\xi-$ and $\eta-$ material lines are orthogonal throughout the process of deformation. Then, it follows from the geometry of Fig.2 that

$$\partial x/\partial\xi = h_\xi\cos\psi,\quad \partial x/\partial\eta = -h_\eta\sin\psi,\quad \partial y/\partial\xi = h_\xi\sin\psi,\quad \partial y/\partial\eta = h_\eta\cos\psi. \tag{4}$$

The compatibility equations are

$$\frac{\partial^2 x}{\partial\xi\partial\eta} = \frac{\partial^2 x}{\partial\eta\partial\xi},\quad \frac{\partial^2 y}{\partial\xi\partial\eta} = \frac{\partial^2 y}{\partial\eta\partial\xi}. \tag{5}$$

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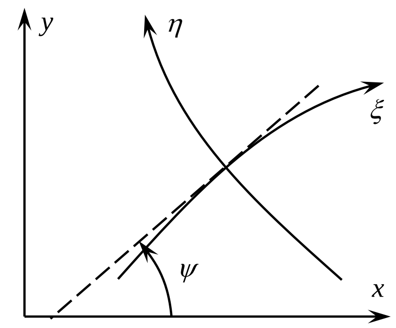

Figure 2. Cartezian (*x*,*y*) and principal lines-based (*ξ*,*η*) coordinate systems.

Substituting (4) into (5) yields

$$\begin{aligned}\cos\psi\,\partial h_\xi/\partial\eta - h_\xi\sin\psi\,\partial\psi/\partial\eta &= -\sin\psi\,\partial h_\eta/\partial\xi - h_\eta\cos\psi\,\partial\psi/\partial\xi,\\ \sin\psi\,\partial h_\xi/\partial\eta + h_\xi\cos\psi\,\partial\psi/\partial\eta &= \cos\psi\,\partial h_\eta/\partial\xi - h_\eta\sin\psi\,\partial\psi/\partial\xi.\end{aligned} \tag{6}$$

It is always possible to choose the Cartesian coordinate system such that $\psi = 0$ at a given point. Moreover, h_η can be eliminated by means of (3). Then, equation (6) becomes

$$h\partial h/\partial\eta + \partial\psi/\partial\xi = 0, \quad h^3\partial\psi/\partial\eta + \partial h/\partial\xi = 0 \tag{7}$$

where $h \equiv h_\xi$. Using a standard technique it is possible to show that this is a hyperbolic system of equations. Its solution can be found independently of the stress equations if appropriate boundary conditions are prescribed. Therefore, it remains to show that the stress equations are compatible with (7). The system of stress equations comprises the equilibrium equations and yield criterion (2). Since the shear stress vanishes in the (ξ,η) coordinate system, the equilibrium equations are [10]

$$h_\eta\,\partial\sigma_\xi/\partial\xi + (\sigma_\xi - \sigma_\eta)\partial h_\eta/\partial\xi = 0, \quad h_\xi\,\partial\sigma_\eta/\partial\eta + (\sigma_\eta - \sigma_\xi)\partial h_\xi/\partial\eta = 0. \tag{8}$$

Equations (2), (3) and (8) combine to give $h\partial\sigma_\xi/\partial\xi - K\,\partial h/\partial\xi = 0$ and $h\partial\sigma_\eta/\partial\eta - K\,\partial h/\partial\eta = 0$. These equations can be immediately integrated to arrive at

$$\sigma_\xi/K - \ln h = f_1(\eta), \quad \sigma_\eta/K - \ln h = f_2(\xi). \tag{9}$$

Here $f_1(\eta)$ is independent of ξ and $f_2(\xi)$ is independent of η. Eliminating $\ln h$ in (9) leads to

$$(\sigma_\xi - \sigma_\eta)/K = f_1(\eta) - f_2(\xi). \tag{10}$$

It is evident that (10) is compatible with (2) if $f_1(\eta) - f_2(\xi) = 1$. Therefore, $f_1(\eta) = C$ and $f_2(\xi) = 1 + C$ where *C* is independent of both ξ and η. Then, equation (10) becomes

$$\sigma_\xi/K - \ln h = C, \quad \sigma_\eta/K - \ln h = C + 1. \tag{11}$$

This equation connects the kinematic quantity h and the stress field in ideal flows.

4. Conclusion
It has been shown that non-trivial nonsteady planar ideal flow solutions exist in anisotropic plasticity assuming that the constitutive equation proposed in [7] is valid. An additional requirement, as compared to ideal flow in isotropic plasticity, is that the principal axes of anisotropy coincide with principal stress trajectories at the initial instant. If this condition is satisfied then equation (11) connects the kinematic quantity h and the stress field. The theory developed is important for training of high skill professionals in Russia [11].

5. Acknowledgment
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