

Effective antiplane shear moduli of the composites with doubly periodic fibers

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Abstract. A novel method for determining the elastic field in the composite with doubly periodic fibers under far-field antiplane strains is presented. The composite is replaced by a homogeneous medium with doubly periodic stresses, which are not related to the strains, in the regions corresponding to the fibers of the composite, the equivalence condition between the composite and homogeneous medium is established. The homogeneous medium with the doubly periodic stresses is solved, and the elastic fields are obtained in the doubly periodic fibers and the matrix. The obtained elastic field is used to evaluate the effective antiplane shear moduli of the composites, good agreements with the existing results are observed.

1. Introduction

Due to weight reducing, strengthening, better fatigue strength and improved corrosion resistance compared to conventional materials, fiber reinforced composites are widely-used in engineering. The composites are heterogeneous medium, which can be seen as homogeneous mediums in structural analysis, it is great important to design and evaluate the effective material properties of such composites. The composite model with periodically distributed reinforcements is certified to be an excellent model for the simulation of effective properties of the composites [1]. Different analytical methods have been developed to study composites with periodic microstructures, such as the Fourier series expansion method [2], the asymptotic homogenization method [3,4], eigenfunction expansion-variational method [5]. Also, the numerical methods such as the boundary element method [6] and the finite element method [7,8] are effective methods to study the composites with periodic microstructures. This paper presents a novel method for determining the elastic field of the composites containing doubly periodic fibers under far-field antiplane strains. The effective antiplane shear moduli of the composites are evaluated and compared with the existing results.

2. Statement of the problem

As shown in figure 1, the parallelogram P_{00} denotes the fundamental cell of the periodic composites, there is a circular cross-section fiber with radius R in the P_{00} , the fiber and the matrix are assumed to be perfectly bonded, the boundary of the P_{00} is denoted by $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, Ω_f^0 and Ω_m^0 denote the regions occupied by the fiber bounded by the contour L_0 and matrix, respectively. The fundamental cell P_{00} is periodically arranged in the complex plane $z = x_1 + ix_2$, the cross-section of the



composite is also shown in figure 1. Let Ω_f and Ω_m denote the union of Ω_f^0 and Ω_m^0 and its periodic congruent regions respectively, L denotes the union of L_0 and its periodic congruent contours. $2\omega_1$ and $2\omega_2$ are two fundamental periods, the vertices of the P_{00} are $\omega_1 + \omega_2$, $-\omega_1 + \omega_2$, $-\omega_1 - \omega_2$ and $\omega_1 - \omega_2$, respectively. The shear moduli of the matrix and the fibers are G_m and G_f , respectively. The composite is subjected to the far-field antiplane strains γ_{13}^∞ and γ_{23}^∞ .

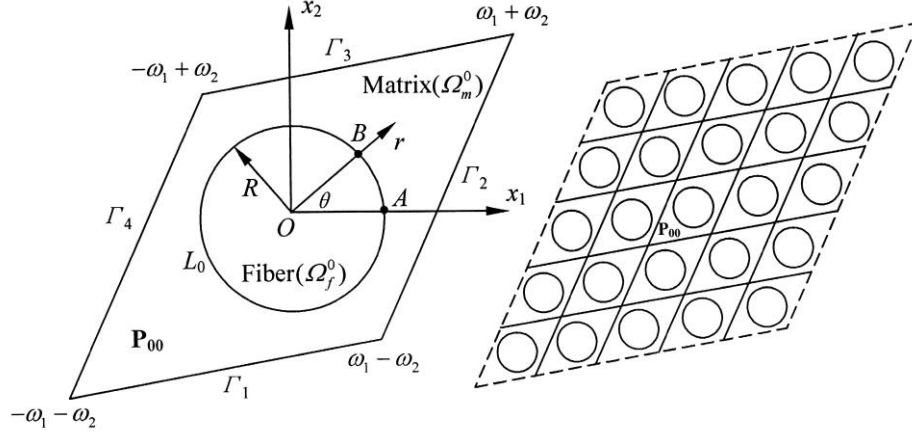


Figure 1. Fundamental cell P_{00} and cross section of the composites

3. Homogenization of heterogeneous medium

For the antiplane problem, there are only the non-trivial antiplane displacement w , strain components γ_{13} and γ_{23} , stress components τ_{13} and τ_{23} , and all field quantities being only the functions of coordinates x_1 and x_2 . Let $\tau = \tau_{13} - i\tau_{23}$ be complex stress, and $\gamma = \gamma_{13} - i\gamma_{23}$ be complex strain.

The original problem: an infinite homogeneous medium with the shear modulus G_m subjected to a far-field antiplane strain $\gamma^\infty = \gamma_{13}^\infty - i\gamma_{23}^\infty$. Apparently, in the entire plane, the strain field γ^0 and the stress field τ^0 are uniform, and given by $\gamma^0 = \gamma^\infty$ and $\tau^0 = G_m \gamma^0 = G_m \gamma^\infty$. For an actual heterogeneous medium under consideration, the presence of the doubly periodic fibers disturbs the uniform strain and stress field. Denote the disturbance strains and stresses by γ'_f and τ'_f in Ω_f and γ'_m and τ'_m in Ω_m respectively, the total stresses and strains in Ω_f and Ω_m are

$$\tau^0 + \tau'_f = G_m \gamma^\infty + \tau'_f \quad \text{in } \Omega_f \quad (1)$$

$$\gamma^0 + \gamma'_f = (\tau^0 + \tau'_f) / G_f = (G_m \gamma^\infty + \tau'_f) / G_f \quad \text{in } \Omega_f \quad (2)$$

$$\tau^0 + \tau'_m = G_m \gamma^\infty + \tau'_m \quad \text{in } \Omega_m \quad (3)$$

$$\gamma^0 + \gamma'_m = (\tau^0 + \tau'_m) / G_m = (G_m \gamma^\infty + \tau'_m) / G_m \quad \text{in } \Omega_m \quad (4)$$

The equivalent problem: an infinite homogeneous medium with the shear modulus G_m subjected to a far-field antiplane strain $\gamma^\infty = \gamma_{13}^\infty - i\gamma_{23}^\infty$. In the entire plane, the strain field γ^0 and the stress field τ^0 are $\gamma^0 = \gamma^\infty$ and $\tau^0 = G_m \gamma^0 = G_m \gamma^\infty$ respectively. Instead of dealing with the presence of the doubly periodic fibers in the original problem, doubly periodic stresses $\tau^* = \tau_{13}^* - i\tau_{23}^*$, which are not related to the strains, are introduced in Ω_f . The introduction of the stresses disturbs the uniform strain and stress fields, with the disturbance strains and stresses being denoted by γ_{in}^* and τ_{in}^* in Ω_f and γ_{out}^* and τ_{out}^* in Ω_m , the total stresses and strains are

$$\tau^0 + \tau^* + \tau_{in}^* = G_m \gamma^\infty + \tau^* + \tau_{in}^* \quad \text{in } \Omega_f \quad (5)$$

$$\gamma^0 + \gamma_{in}^* = (G_m \gamma^\infty + \tau^* + \tau_{in}^* - \tau^*) / G_m = (G_m \gamma^\infty + \tau_{in}^*) / G_m \quad \text{in } \Omega_f \quad (6)$$

$$\tau^0 + \tau_{out}^* = G_m \gamma^\infty + \tau_{out}^* \quad \text{in } \Omega_m \quad (7)$$

$$\gamma^0 + \gamma_{out}^* = (\tau^0 + \tau_{out}^*) / G_m = (G_m \gamma^\infty + \tau_{out}^*) / G_m \quad \text{in } \Omega_m \quad (8)$$

The equivalence between the two problems requires the same stresses and strains, let

$$\tau^* + \tau_{in}^* = \tau_f' \quad (9)$$

$$\tau_{out}^* = \tau_m' \quad (10)$$

and from equations (1), (5) and (3), (7), the stress equivalence is arrived at. A comparison of equations (4) and (8) shows that the strain equivalence in Ω_m is also arrived at. From equations (2) and (6), the strain equivalence in Ω_f results in the following equation

$$\left(\frac{G_m}{G_f} - 1\right) \gamma^\infty + \left(\frac{1}{G_f} - \frac{1}{G_m}\right) \tau_{in}^* + \frac{1}{G_f} \tau^* = 0 \quad (11)$$

4. Elastic field induced by the stress τ^*

The analytical function $F(z)$ can be used to formulate the antiplane problem:

$$w = F(z) + \overline{F(z)} \quad (12)$$

$$\tau = G_m \gamma = 2G_m dF(z)/dz \quad (13)$$

$$T = \int_A^B \tau_{nz} ds = \int_A^B (\tau_{13} dx_2 - \tau_{23} dx_1) = G_m i [\overline{F(z)} - F(z)]_A^B \quad (14)$$

where T is the resultant shear stress on an arc AB , $[\cdot]_A^B$ signifies the change in the bracketed function in moving from point A to point B along the arc AB , the overbar denotes the conjugate.

To describe the stress τ^* , a function $F^*(z)$ is introduced and expanded into Taylor series in Ω_f^0

$$F^*(z) = \sum_{k=1}^{\infty} A_k z^k \quad z \in \Omega_f^0 \quad (15)$$

From equations (13) and (15), the stress τ^* can be expressed as

$$\tau^* = \tau_{13}^*(z) - i \tau_{23}^*(z) = 2G_m \frac{dF^*(z)}{dz} = 2G_m \sum_{k=1}^{\infty} k A_k z^{k-1} \quad z \in \Omega_f^0 \quad (16)$$

The stress on boundary L_0 in polar coordinates (r, θ) can be expressed as

$$\tau_{r3}^*(t) - i \tau_{\theta 3}^*(t) = e^{i\theta} [\tau_{13}^*(t) - i \tau_{23}^*(t)] = \frac{2G_m}{R} \sum_{k=1}^{\infty} k A_k t^k \quad t \in L_0 \quad (17)$$

The interfacial stress $\tau_{r3}^*(t)$ on L_0 is

$$\tau_{r3}^*(t) = \frac{[\tau_{r3}^*(t) - i \tau_{\theta 3}^*(t)] + \overline{[\tau_{r3}^*(t) - i \tau_{\theta 3}^*(t)]}}{2} = \frac{G_m}{R} \left(\sum_{k=1}^{\infty} k A_k t^k + \sum_{k=1}^{\infty} k \bar{A}_k R^{2k} \frac{1}{t^k} \right) \quad t \in L_0 \quad (18)$$

Selecting arc AB on L_0 as shown in figure 1, A is a fixed point on x_1 axis and B is a moving point. The resultant shear stress on arc AB can be written as

$$T^*(t) = \int_A^B \tau_{r3}^*(t) ds \quad t \in L_0 \quad (19)$$

The substitution of equation (18) into equation (19) yields

$$T^*(t) = \frac{G_m}{i} \sum_{k=1}^{\infty} [(A_k t^k - \bar{A}_k R^{2k} \frac{1}{t^k}) - R^k (A_k - \bar{A}_k)] \quad t \in L_0 \quad (20)$$

Let $F(z)$ denotes the complex potential function of the elastic field induced by the stress τ^* . Owing to the double periodicity of the stress τ^* , the stresses and strains induced by τ^* are also doubly periodic, the displacement field induced by τ^* and the function $F(z)$ are doubly quasi-periodic.

The continuity conditions of the displacement on L can be written as

$$w^+(t) - w^-(t) = 0 \quad t \in L \quad (21)$$

from equation (12), equation (21) can be rewritten as

$$[F^+(t) + \overline{F^+(t)}] - [F^-(t) + \overline{F^-(t)}] = 0 \quad t \in L \quad (22)$$

The jump conditions of the resultant on L can be written as

$$T^+(t) - T^-(t) = -T^*(t) \quad t \in L \quad (23)$$

from equation (14), equation (23) can be rewritten as

$$[F^+(t) - \overline{F^+(t)}] - [F^-(t) - \overline{F^-(t)}] = \frac{T^*(t)}{G_m i} \quad t \in L \quad (24)$$

where the superscripts “+” and “-” signify the boundary values of the quantities as approached from the interior and the exterior regions of the contour L , respectively.

From equations (22) and (24), it is seen that

$$F^+(t) - F^-(t) = \frac{T^*(t)}{2G_m i} \quad t \in L \quad (25)$$

The solution of equation (25) (a doubly quasi-periodic Riemann boundary problem) in the P_{00} is

$$F(z) = C_0 z + \frac{1}{2\pi i} \int_{L_0} \frac{T^*(t)}{2G_m i} \zeta(t-z) dt \quad z \in \mathcal{Q}_f^0 + \mathcal{Q}_m^0 \quad (26)$$

where $\zeta(\cdot)$ is Weierstrass Zeta function, C_0 is the constant to be determined.

Although $F(z)$ is solved in the fundamental cell P_{00} , it only differs by an addend in the neighboring unit cell, which is not related to the stress and strain. Substituting equation (20) into equation (26), $F(z)$ can be determined as following

$$F(z) = C_0 z - \frac{1}{2} \sum_{k=1}^{\infty} \left\{ A_k z^k - \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \left[\zeta(z) - \frac{1}{z} \right]^{(k-1)} - R^k (A_k - \bar{A}_k) \right\} \quad z \in \mathcal{Q}_f^0 \quad (27)$$

$$F(z) = C_0 z + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \zeta^{(k-1)}(z) \quad z \in \mathcal{Q}_m^0 \quad (28)$$

then the strains and stresses induced by the stress τ^* are derived from equation (13)

$$\gamma_{in}^* = 2C_0 - \sum_{k=1}^{\infty} \left\{ kA_k z^{k-1} - \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \left[\zeta(z) - \frac{1}{z} \right]^{(k)} \right\} \quad z \in \Omega_f^0 \quad (29)$$

$$\gamma_{out}^* = 2C_0 + \sum_{k=1}^{\infty} \left\{ \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \zeta^{(k)}(z) \right\} \quad z \in \Omega_m^0 \quad (30)$$

$$\tau_{in}^* = 2G_m C_0 - \sum_{k=1}^{\infty} G_m \left\{ kA_k z^{k-1} - \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \left[\zeta(z) - \frac{1}{z} \right]^{(k)} \right\} \quad z \in \Omega_f^0 \quad (31)$$

$$\tau_{out}^* = 2G_m C_0 + \sum_{k=1}^{\infty} G_m \left\{ \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \zeta^{(k)}(z) \right\} \quad z \in \Omega_m^0 \quad (32)$$

For far-field antiplane strain γ^∞ , owing to the periodicity, the difference of the disturbed displacement between the two endpoints of each boundary Γ_k is zero. From equation (12), it follows that

$$[\overline{F(z)} + F(z)]_{\Gamma_k} = 0 \quad (33)$$

The substitution of equation (28) into equation (33) yields

$$C_0 = \frac{\pi R^2}{2S} (A_1 + \delta_2 \bar{A}_1) \quad (34)$$

where $S = 2i(\omega_1 \bar{\omega}_2 - \omega_2 \bar{\omega}_1)$ is the area of the P_{00} , $\delta_2 = 2(\bar{\omega}_1 \eta_2 - \bar{\omega}_2 \eta_1) / (\pi i)$, $\eta_1 = \zeta(\omega_1)$, $\eta_2 = \zeta(\omega_2)$.

5. Stresses and strains in the fibers and the matrix

Noting equations (9) and (10), from equations (1-4), the total stresses and strains are

$$\begin{aligned} \tau_f = G_m \gamma^\infty + \tau^* + \tau_{in}^* = G_m \gamma^\infty + 2G_m \sum_{k=1}^{\infty} kA_k z^{k-1} + \\ G_m \left\{ \frac{\pi R^2}{S} (A_1 + \delta_2 \bar{A}_1) - \sum_{k=1}^{\infty} \left[kA_k z^{k-1} - \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \left[\zeta(z) - \frac{1}{z} \right]^{(k)} \right] \right\} \end{aligned} \quad z \in \Omega_f^0 \quad (35)$$

$$\gamma_f = \tau_f / G_f \quad z \in \Omega_f^0 \quad (36)$$

$$\tau_m = G_m \gamma^\infty + \tau_{out}^* = G_m \gamma^\infty + G_m \left\{ \frac{\pi R^2}{S} (A_1 + \delta_2 \bar{A}_1) + \sum_{k=1}^{\infty} \left[\frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \zeta^{(k)}(z) \right] \right\} \quad z \in \Omega_m^0 \quad (37)$$

$$\gamma_m = \tau_m / G_m \quad z \in \Omega_m^0 \quad (38)$$

Substituting equations (16) and (31) in equation (11), the equivalent condition can be written as

$$\begin{aligned} \left(\frac{G_m}{G_f} - 1 \right) \gamma^\infty + \left(\frac{G_m}{G_f} - 1 \right) \left\{ \frac{\pi R^2}{S} (A_1 + \delta_2 \bar{A}_1) - \sum_{k=1}^{\infty} \left[kA_k z^{k-1} - \frac{(-1)^k}{(k-1)!} \bar{A}_k R^{2k} \left[\zeta(z) - \frac{1}{z} \right]^{(k)} \right] \right\} + \\ 2 \frac{G_m}{G_f} \sum_{k=1}^{\infty} kA_k z^{k-1} = 0 \end{aligned} \quad (39)$$

From equation (39), the constants A_k ($k=1,2,3,\dots$) can be uniquely determined, then the stresses and strains in the doubly periodic fibers and the matrix can be obtained from equations (35-38).

6. Effective antiplane moduli of composites

According to the average strain theorem of elasticity, for a two-phase composite under far-field antiplane strain $\gamma^\infty = \gamma_{13}^\infty - i\gamma_{23}^\infty$, the effective antiplane moduli are determined by the following equation

$$\begin{bmatrix} G_m - C_{44}^{eff} & -C_{45}^{eff} \\ -C_{45}^{eff} & G_m - C_{55}^{eff} \end{bmatrix} \begin{Bmatrix} \gamma_{23}^\infty \\ \gamma_{13}^\infty \end{Bmatrix} = \lambda \begin{bmatrix} G_m - G_f & 0 \\ 0 & G_m - G_f \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{23,f} \\ \bar{\gamma}_{13,f} \end{Bmatrix} \quad (40)$$

where λ is fiber volume fraction, $\bar{\gamma}_{13,f}$ and $\bar{\gamma}_{23,f}$ are the averaged antiplane strains.

Fiber square arrangement bring out transverse isotropy, i.e., $C_{44}^{eff} = C_{55}^{eff} = G^{eff}$, $C_{45}^{eff} = 0$, the effective antiplane shear moduli of the square fiber arrangement was studied by Fourier series expansion method [2], a comparison with literature [2] is shown in table 1, good agreements are observed.

Table 1. A comparison with the Fourier series expansion method

λ	G_f/G_m	G^{eff}/G_m	
		literature [2]	present
0.4	6	1.805	1.80451
	20	2.147	2.14586
	120	2.314	2.31343
0.55	6	2.326	2.32562
	20	3.184	3.07712
	120	3.555	3.50663
0.7	6	3.176	3.17311
	20	5.222	5.21367
	120	6.945	6.92944
0.75	6	3.620	3.61966
	20	7.006	7.00409
	120	11.170	11.1640

7. Conclusions

Using the equivalent inclusion technique integrated with the result of doubly periodic Riemann boundary value problem, an analytic method for determining the elastic field of the composites with doubly periodic fibers under far-field antiplane strains is developed. The solution is used to evaluate the effective antiplane moduli of the composites, comparisons with the existing results show the correctness of the present solution. The present method can provide benchmark results for other methods, and may be useful in the computer-aid design of new materials.

References

- [1] Trias D, Costa J, Mayugo J A and Hurtado J E 2006 *Comp. Mater. Sci.* **38** 316–24
- [2] Chen C H 1970 *J. Appl. Mech. –T. Asme* **37** 198–201
- [3] Rodriguez-Ramos R, Sabina F J, Guinovart-Diaz R and Bravo-Castillero J 2001 *Mech. Mater.* **33** 223–35
- [4] Andrianov I V, Danishevs'kyy V V and Kalamkarov A L 2008 *Composites Part B* **39** 874–81
- [5] Yan P, Jiang C P and Song F 2011 *Mech. Mater.* **43** 586–97
- [6] Dong C Y 2006 *Int. J. Solids Struct.* **43** 7919–38
- [7] Xia Z H, Zhang Y F and Ellyin F 2003 *Int. J. Solids Struct.* **40** 1907–21
- [8] Xu Y L, Du S S, Xiao J H and Zhao Q X 2012 *Comp. Mater. Sci.* **61** 34–41