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γ - Uniquely colorable graphs

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Abstract. A graph G = (V, E) is uniquely colorable if the chromatic number χ (G) = n and every n – coloring of G induces the same partition of V. In this paper, we introduce a new kind of graph called γ - uniquely colorable graphs. We obtain a necessary and sufficient condition for a graph to be γ - uniquely colorable graphs. We provide a constructive characterization of γ - uniquely colorable trees.

1. Introduction

In [1] Benedict Michael Raj et al., studied a few properties of two invariants, dcc(G) and dccs(G). In [2], John Arul Singh and Kala investigated graphs with md χ(G) = 0 and also proved certain if and only if conditions such that md χ(G) = χ (G). In [3] Benedict Michael Raj et al obtained some bounds for the chromatic transversal domatic number, d_{ct} (G) and characterized graphs attaining the bounds. Also, characterized uniquely colorable graphs with d_{ct}(G) = 1. Finally obtained Nordhaus–Gaddum inequalities for d_{ct}(G) and characterized graphs for which $d_{ct}(G) + d_{ct}(\overline{G}) = p$ and $p - 1$. In [4] Michael Dorfling et al provided a simple constructive characterization for trees. In [5] David E. Brown et al characterized the class of 2-trees which are interval 3- graphs.

2. Terminology

We consider only simple connected undirected graphs G = (V, E) with n vertices and m edges. H is a subgraph of G, if vertex set of H is contained in vertex set of G and (uv)$\in$ E (H) implies (uv)$\in$ E (G). A subgraph H is said to be an induced subgraph of G if for every pair u, v of vertices, (uv)$\in$ E(H) implies (uv)$\in$ E (G) and is denoted by $\langle$ H $\rangle$. A path is a trail in which all vertices (except perhaps the first and last ones) are distinct, P_n denotes the path with n vertices. A cycle is a circuit in which no vertex except the first (which is also the last) appears more than once. C_n is a cycle with n vertices. K_n is a complete graph with n vertices. For properties related to graph theory, we refer to F. Harary [6]. Given a simple, connected graph G, partition all vertices of G into a smaller possible number of disjoint, independent sets. This is known as the chromatic partitioning of graphs.

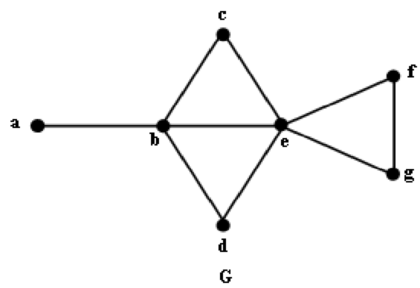

Figure 1.

A graph G = (V, E) is uniquely colorable if the chromatic number χ (G) = n and every n – coloring of G induces the same partition of V.

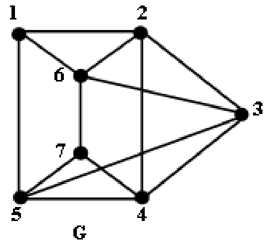

Figure 2.

A set of vertices D in G is a dominating set if every vertex of V – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set – abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by γ (G).The private neighborhood of v $\in$ D is defined by pn [v, D] = N (v) – N (D – { v}). For properties related to domination, we refer to T. W. Haynes, S. T. Hedetniemi, and P. J. Slater [7].

3. Results and Discussions

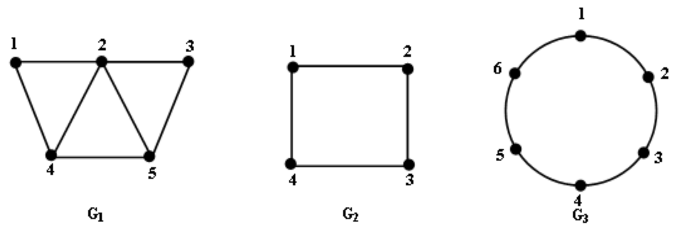

Figure 3.

In Fig. 3 G_1, G_2 and G_3 are uniquely colorable graphs, with chromatic partition P_1 = {{ 2 }, { 3, 4 }, { 1, 5 }}, P_2 = { {1, 3 }, { 2, 4 } } and P_3 = {{ 1, 3, 5 }, { 2, 4, 6 } } . We observe that in P_1, { 2 } is a γ - set for G_1, while in P_2 every set in the partition is a γ - set and in P_3 the partition has no γ - set. So we understand that, there are uniquely colorable graphs where at least one set in the partition is a γ - set. We restrict onto uniquely colorable graphs whose chromatic partition contains atleast one γ - set. We call such graphs as γ - uniquely colorable graphs and the chromatic partition of such graphs as γ - chromatic partition.

Theorem 1

Let G be a uniquely colorable graph. Let P be the chromatic partition for G. Let D be an independent γ - set for G. D $\in$ P if and only if there exist a partition P_1 of V – D such that

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042115 doi:10.1088/1757-899X/263/4/042115

1. P_1 is unique
2. every set in P_1 is independent
3. $| P_1 | = k – 1$ where $| P | = k$.

Proof

Let G be a uniquely colorable graph. Let P be the chromatic partition for g. Let D be a γ - set for G. Let $| P | = k$. Assume that $D \in P$. Let $P = \{ x_1, x_2, \ldots, x_k \}$. Let $D \in x_i$. Any vertex in $x_j \in V – D$, j = 1 to k, $i \neq j$. Since P is the chromatic partition for G, $P_1 = \{ x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \}$ is a partition for V – D such that

1. P_1 is unique
2. every set in P_1 is independent
3. $| P_1 | = k – 1$ where $| P | = k$.

Conversely, assume that there exists a partition P_1 for V – D satisfying the conditions of the theorem. Let $P_1 = \{ x_1, x_2, \ldots, x_{k-1} \}$. Let $P = P_1 \cup \{ D \}$.

1. $\bigcap_{i=1}^{k} x_i = \phi$
2. $P_1 \cap D = \phi$
3. $P_1 \cup D = V (G)$
4. x_i, D, i = 1 to k – 1 are independent.

Hence P is a chromatic partition for G.

Remark

G has a chromatic partition P not containing any γ - set if and only if either

1. G has no independent γ - set
2. If G has an independent γ - set then conditions of Theorem 1 fails.

Proof

Let P be the chromatic partition not containing any γ - set of G., In this case, it is obvious that

1. G has no independent γ - set or
2. If G has an independent γ - set then there exists no partition P_1 of V – D satisfying the conditions of Theorem 1(else if a partition exists for V – D then the assumption that P does not contain any γ - set fails).

Conversely, if the conditions of the remark satisfied, then P has no γ - set.

Theorem 2

If P is the chromatic partition for a uniquely colorable graph G, then every set in P is a dominating set.

Proof

Let $P = \{ x_1, x_2, \ldots, x_k \}$ be a chromatic partition for G. Assume that there exist some x_i, i = 1 to k such that x_i is not a dominating set then $\exists$at least one vertex $u \in V (G)$, $u \notin x_i$, $\ni$u not adjacent to any vertex in x_i. Assume that $u \in x_j$, $j \neq i$, $P_1 = \{ x_1, x_2, \ldots, x_{i-1}, x_i \cup \{ u \}, x_{i+1}, \ldots, x_{j-1}, x_j \cup \{ u \} , x_{j+1}, \ldots, x_k \}$ is a chromatic partition for G, a contradiction for our assumption that G is uniquely colorable.

Theorem 3

Let G be a uniquely colorable graph $| P | = 2$ if only if $N (u) \in V – D \ \forall u \in D$, $N (w) \in D \ \forall w \in V – D$.

Proof

Let G be uniquely colorable and $| p | = 2 = \{ x_1, x_2 \}$ (say). If for some $u \in D \ \exists$ a vertex $v \in V – D \ni v \in D$ then $P = \{ x_1, x_2 \}$ is a partition for G such that u, v belongs to some X_i, i = 1, 2 , a contradiction to our assumption on P.

Similarly, if for some $w \in V – D$ there exist some $w \in V– D$ there exist some $x \in V – D$ such that $x \in N (w)$ then $w, x \in V – D$, w adjacent to v, w, v belongs to some x_i, i = 1, 2, a contradiction to our assumption on P.

Conversely, assume that for every $u \in D$, $N (u) \in V – D$ for all $w \in D$, $N (w) \in V – D$. If possible, assume that $| P | = 3 = \{ x_1, x_2 , x_3 \}$ (say). Let one of x_i, i = 1, 2, 3 be a γ - set for G. Let $x_1 = D$, this

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042115 doi:10.1088/1757-899X/263/4/042115

means that x_2, $x_3 \in V - D$. By our assumption there exist no $x, y \in V - D$, $x \perp y$. So $x_2 \cup x_3$ is an independent set, which implies $P_1 = \{ x_1, x_2 \cup_{x3} \}$ is a partition for G such that

1. x_1, $x_2 \cup x_3$ are independent
2. x_1 is a γ - set for G
3. $x_2 \cup x_3 \in V - D$

That is, P_1 is a chromatic partition for G such that $| P_1 | < | P |$, a contradiction to our assumption that P is a chromatic partition for G.

Remark

1. For any tree, we know that $| P | = 2$, so if P is chromatic partition for T such that $| P| = 2 = \{ x_1, x_2 \}$ at least one of xi is a γ- set for T then by the above theorem, we conclude that the following statement
   T is a uniquely colorable tree if and only if for $u \in D$, $N (u) \in V - D$, $\forall w \in V - D$, $N(w) \in D$.

By the Theorem 3, we conclude that

R1: If T is a uniquely colorable tree then

1. every internal vertex is two dominated
2. if a pendant vertex $u \in D$, then for the support vertex v adjacent to u, u is the only leaf.

Theorem 4

If T is a uniquely colorable tree then $\gamma (T) + \gamma (\overline{T}) = \gamma (T) + 2$

proof

Since T has atleast two pendant vertices u_1, u_2 (say). In$\overline{T}$, u_1 dominates $V (\overline{T}) - N (u_1)$. $N (u_1)$ is dominated by u_2 implies $\gamma (\overline{T}) = 2$.

3.1. Trees

Theorem 5

Let T be a γ - uniquely colorable tree. Let $P = \{V_1,V_2\}$ be a γ - chromatic partition for T. H is generated from T by attaching a path P_1 at u where $u \in V (G)$. Let $\gamma (H) = \gamma (T)$. H is γ - uniquely colorable if and only if $u \in V_1$.

Proof

Assume that H is γ - uniquely colorable tree. There exist a γ - chromatic partition P_1 for H such that P1 $= \{V_1, V_2\}$. Let D_1 be a γ - set for H and D be a γ - uniquely colorable γ - set for T. By assumption, $| D_1 | = | D |$. Let v be a new pendant vertex attach at u to generate H. Either $v \in V_1$ or $u \in V_1$. If $v \in V_1$, then $D_1 - \{ v \}$ is a γ - set for T such that $| D | > | D_1 - \{v\} |$, a contradiction to our assumption that D is a γ - set for T, implies $u \in V_1$. $P_2 = \{ P_1 - \{v\}\} = \{V_1, V_2 - \{v\}\}$ is a γ - chromatic partition for T. $P_2 = P$ since T is γ - uniquely colorable tree.

Conversely, assume that $u \in V$, we have to prove that H is γ - uniquely colorable tree. D is a γ - set for H and $P_3 = \{P \cup \{v\}\} = \{ V_1, V_2 \cup \{v\}\}$ is a chromatic partition for H such that

1. $V_1 \in D$.
2. $V_2 \in V - D$.
3. $N (u) \in V - D$, for all $u \in D$.
4. $N (w) \in D$, for all $w \in V - D$.

implies H is γ - uniquely colorable tree.

Note

Theorem 5 states that, H isγ - uniquely colorable tree if and only if $u \in V_1$. If u is any vertex in H which is a good vertex but $u \notin V_1$, then the resulting graph H need not be uniquely colorable. For example, consider the graph G in Figure 4

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042115 doi:10.1088/1757-899X/263/4/042115

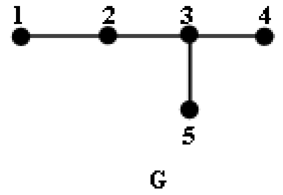

Figure 4.

G is uniquely colorable with a γ - chromatic partition P = { { 1, 3 }, { 2, 4, 5 } }. { 2, 3 } is also a γ-set for G. Attaching a path of length 1 at vertex 2 results to the graph seen in Fig. 5 which is not uniquely colorable.

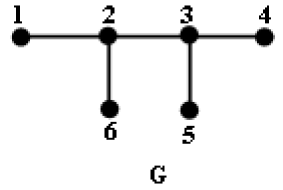

Figure 5.

Theorem 6

Let T be a γ - uniquely colorable tree. Let P = { V_1, V_2 } be a γ - chromatic partition for T. H is generated from T by attaching a path P_1 at u where u $\in$ V (G). Let γ (H) = γ (T) + 1 . H is γ - uniquely colorable if and only if u $\in$ bad.

Proof

Assume that H is γ - uniquely colorable tree. There exist a γ - chromatic partition P_1 for H such that P_1 = { V_1, V_2 }. Let D_1 be a γ - set for H and D be a γ - uniquely colorable γ - set for T. Let v be a new pendant vertex attach at u to generate. u is a bad vertex with the vertex to T else if u is good with respect to T, then D itself is a γ - set for H, a contradiction to our assumption that γ (H) = γ (T) + 1 .

Conversely, assume that u is bad with respect to T. Since γ (H) = γ (T) + 1 , let D_1 = D $\cup$ { v } be a γ - set for H. Since T is γ - uniquely colorable, P_1 = P $\cup$ { v } = { $V_1 \cup$ { v }, V_2 } = { V_3, V_2 } is a chromatic polynomial for H such that

1. v $\in$ V_3
2. u $\in$ V_2
3. N (u) $\in$ V – D_1, for all u$\in$ D_1
4. N (w) $\in$ D_1, for all w $\in$ V – D_1

implies D_1 is a γ - uniquely colorable γ - set for H and hence H is γ - uniquely colorable tree.

Theorem 7

Let T be a γ - uniquely colorable tree. Let P = { V_1, V_2 } be a γ - chromatic partition for T. H is generated from T by attaching a path P_2 at u where u $\in$ V (G). Let γ (H) = γ (T) + 1. H is γ - uniquely colorable if and only if u is not selfish with respect to T.

Proof

Assume that H is γ - uniquely colorable tree. H is generated from T by attaching a path P_2 to u. There exist a γ - chromatic partition P_1 for H such that P_1 = { V_1, V_2 }. Let D_1 be a γ - set for H and D be a γ - uniquely colorable γ - set for T. Let the vertex adjacent to u be v and w be the vertex adjacent to v. If possible, assume that u is selfish with respect to T. Then D_1 = D – { u } $\cup$ { v } is a γ - set for H. Since H is uniquely colorable there exist a γ - uniquely colorable γ - set D_2 for H and a γ - chromatic partition P_1 = { V_3, V_4 } for H. Either v $\in$ V_3 or w $\in$ V_3 (since w is pendant). D_3 = D_2 – { v } is a γ-set for T such that | D_3 | < | D_2 | , a contradiction to our assumption that D_2 is a γ- set for H. If v $\in$ V_3, then u $\in$ V – D_2 and u is 2 – dominated with respect to D_2. If w $\in$ V_3, then v $\in$ V – D_2 and u $\in$ V_3. D_3 = D_2 – { w } is a γ - set for T such that | D_3 | < | D_2 |, a contradiction to our assumption that D_2 is a γ - set for H.

Conversely, assume that u is not selfish with respect to T. We know that T is uniquely colorable and P = { V_1, V_2 } is a γ - chromatic partition for T. $D_4 = D \cup \{ w \}$ is a γ - set for H (since $\gamma (H) = \gamma (T) + 1$). Also $P_2 = \{ P \cup \{ w \} \} = \{ V_1 \cup \{ w \}, V_2 \} = \{ V_5, V_2 \}$ is a chromatic partition for H such that

1. N (u) $\in V - D_4$, for all u $\in D_4$ and
2. N (w) $\in$ D, for all w $\in V - D_4$

implies H is uniquely colorable.

3.2.Tree characterization

In this section, we present a constructive characterization of trees T that a γ - uniquely colorable tree.

Operation O_1: Attach a tree path P_1 to a vertex u of T to generate T_1, so that

1. $\gamma (T) = \gamma (T')$
2. u $\in$ D where D is a γ - uniquely colorable γ - set with respect to T.

Operation O_2: Attach a tree path P_1 to a vertex u of T to generate T_1, so that

1. $\gamma (T_1) = \gamma (T) + 1$.
2. u is a bad vertex with respect to T.

Operation O_3: Attach a tree path P_1 to a vertex u of T to generate T_1, so that

1. $\gamma (T_1) = \gamma (T) + 1$.
2. u is not a selfish vertex with respect to T.

Let τ be the family defined by τ = { T / T is obtained from K_1, by a finite sequence of operations O_1 or O_2 or O_3 }.

From Theorem and we know that if T $\in \tau$, then T is a γ - uniquely colorable tree.

Theorem 8

If T is a γ - uniquely colorable tree, then T $\in \tau$.

Proof

We proceed by induction on the order n $\geq$ 1. If T is a star, then T can be generated from K_1, by repeated application of Operation O_1. Hence we may assume that diam (T) $\geq$ 3. Assume that the Theorem is true for all tree T ' of order n' < n. Let T be rooted at a leaf r, of longest path r – u path P. Let v be the neighbor of u. Further, let w denote the parent of v. By T_x , we denote the subtree induced by vertex x and its descendants in the rooted tree T.

Let T ' = T – T_u. Let d_T (v) $\geq$ 4, v is a support vertex with respect to T ' ∍the number if pendant vertices adjacent to v is at least 2. Since T ' is uniquely colorable there exist a γ - uniquely colorable γ - set D_1 for T ' containing v, that is D_1 is aγ - uniquely colorable γ - set for T ' containing v. Also γ (T) = γ (T ') implies T can be obtained from T ' by operation O_1 .

Let d_T (v) = 3. Label the pendant vertex adjacent to v as x. Any tree has a γ - set containing all the pendant vertices. Let D_1 be a γ - set for T ' containing v. D_1 itself is a γ - set for T implies γ (T) = γ (T ') . We know that T is γ - uniquely colorable tree. In T, d (v) = 2, implies T has a γ - uniquely colorable γ - set D ∍ v $\in$ D (R1) . D itself is a γ - uniquely colorable γ - set for T ' i.e., D is a γ - uniquely colorable γ - set for T ' containing v. Also γ (T) = γ (T ') implies T can be generated from T ' by applying operation O_2.

If u$\in$ D, then D_1 = D – { u } is a dominating set for T ' . If T ' has a γ - set D_2 such that | D_2 | < | D_1 |, then

$$D_3 = \begin{cases} D_2 \cup \{ u \} & \text{if } v \notin D_2 \quad \text{or} \\ D_2 & \text{if } v \in D_2 \end{cases}$$

is a γ - set for T ∍| D_3 | < | D |.W have assumed that D is a γ - uniquely colorable γ - set for T. So a γ - set with smaller cardinality is not possible, implies D_1 is a γ - set for T '.

Let d $_T$ (v) = 2. Since T is γ - uniquely colorable there exist a γ - uniquely colorable γ - set for D for T. Either u $\in$ D or v $\in$ D. If v $\in$ D, then since γ (T) = γ (T '), T can be generated from T ' by applying operation O_1. If v is a good vertex with respect to T ', then there exist a γ - set D_4 for T '

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042115 doi:10.1088/1757-899X/263/4/042115

containing v. D is a γ - uniquely colorable γ - set for T. So, a γ - set for T with smaller cardinality is not possible implies D_4 cannot be a γ - set for T, implies v is a bad vertex with respect to T '. Also γ (T) = γ (T ') + 1 , implies T can be generated from T ' by applying operation O_2.

Let T ' = T – T_v. Since T is γ - uniquely colorable and v is a support vertex, any γ - uniquely colorableγ - set D for T contains either u or v. When u $\in$ D, w also belongs to D. When v $\in$ D, w is two dominated, then D_1 = D – { u } or D_1 = D – { u } is a dominating set for T ' . If T ' has a γ - set D_2 such that | D_2 | < | D_1 |, then $D_3 = D_2 \cup$ { v } is a γ - set for T such that | D_3 | < | D |. We have assumed that D is a γ - uniquely colorable γ - set for T. So, a γ - set with smaller cardinality is not possible implies D_1 is a γ - set for T ' that is, γ (T ') = γ (T) – 1. If w is selfish with respect to T ', then D_4 = D_1 – { w } $\cup$ { v } is a γ - set for T $\ni$| D_4 | < | D_1 |, a contradiction to the assumption that γ (T ') = γ (T) – 1 implies w is not selfish with respect to T '. Also γ (T) = γ (T ') + 1 , implies T can be generated from T ' by applying operation O_3 .

As a immediate consequence of Theorems 5, 6 and 7, we have following characterization of γ - uniquely colorable γ - set.

Theorem 9

A tree T is γ - uniquely colorable tree if and only if T $\in\tau$.

4. Conclusion

This paper contributes the necessary and sufficient condition, tree characterization of a γ - uniquely colorable graphs.

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