IOP Conf. Series: Materials Science and Engineering **263** (2017) 042157 doi:10.1088/1757-899X/263/4/042157

Subclass of uniformly convex functions defined by linear operator

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Abstract. Making use of certain linear operator, we define a new subclass of uniformly convex functions with negative coefficients and obtain coefficient estimates, extreme points, closure and inclusion theorems and the radii of star likeness and convexity for the new subclass. Furthermore, results partial sums are discussed.

1.Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $E = \{z : z \in C |z| < 1\}$. Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0) \tag{1.2}$$

Following Goodman [2 and 3], Ronning [4 and 5] introduced and studied the following sub-classes

(i)A function $f \in A$ is said to be in the class $S_p(\alpha)$ uniformly starlike functions if it satisfies the condition.

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in E, \tag{1.3}$$

$-1 < \alpha \leq 1$.

(ii) A function $f \in A$ is said to be in the class $UCV(\alpha)$, uniformly convex functions if it satisfies the condition.

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, z \in E, \tag{1.4}$$

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and $-1<\alpha\leq 1$.
Indeed it follows from (1.3) and (1.4) that

$$f\in UCV(\alpha)\Leftrightarrow zf'\in S_p(\alpha). \tag{1.5}$$

For functions $f\in A$ given by (1.1) and $g(z)\in A$ given by $g(z)=z+\sum_{n=2}^{\infty}b_n z^n$ we define the Hadamard product (or Convolution) of f and g by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_n b_n z^n,\ \mathrm{z}\in\mathrm{E} \tag{1.6}$$

Let $\phi(a;c;z)$ be the incomplete beta function defined by

$$\phi(a;c;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}z^n,\ c\neq 0,-1,-2,... \tag{1.7}$$

Where $(\lambda)_n$ is the Pochhammer symbol defined interms of the Gamma functions, by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & n=0\\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1), & n\in N\end{cases} \tag{1.8}$$

Further, for $f\in A$

$$L(a,c)f(z)=\phi(a;c;z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^n, \tag{1.9}$$

where $L(a,c)$) is called Carlson – Shaffer operator [1] and the operator * stands for the hadamard product (or convolution product) of two power series is given by (1.6). We notice that

$$L(a,a)f(z)=f(z),L(2,1)f(z)=zf'(z)$$

Now, we define a Generalized carson – Shaffer operator $L(a;c:\gamma)$ by

$$L(a;c:\gamma)f(z)=\phi(a;c;z)*D_r f(z) \tag{1.10}$$

For a function $f\in A$ where

$$D_r f(z)=(1-\gamma)f(z)+\gamma\, zf'(z)(n\geq 0z\in E)$$

So, we have

$$L(a;c;r)f(z)=z-\sum_{n=2}^{\infty}[1+(n-1)r]\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^n \tag{1.11}$$

It is easy to observe that for γ=0, we get the Carlson- Shaffer linear operator [1].

For $-1\leq\alpha<1$ we let $S(\alpha,\gamma)$ be the subclass of functions of the form (1.1) and satisfying the analytic criterion.

$$Re\left\{\frac{\mathrm{z}(L(a,c;\gamma)f(z))'}{L(a,c;\gamma)}-\alpha\right\}>\left|\frac{z\left(L(a,c;\gamma)f(z)'\right)}{L(a,c;\gamma)}-1\right|$$

where ($L(a,c;\gamma)f(z)$ we also let (1.11) we also let

$$TS(\alpha,\gamma)=S(\alpha,\gamma)\cap T$$

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By suitably specializing the values of (a) and (c), the class $S(\alpha,\gamma)$ can reduces to the class studied earlier by Ronning [5,6]. Also choosing $\alpha=0$ and $\gamma=1$ the class coincides with the class studied in [11] and [12] respectively.

2. Main Results

Theorem 2.1. A function $f(z)$ of the form (1.1) is in $S(\alpha,\gamma)$ if

$$\sum_{n=2}^{\infty}\left[2n-(\alpha+1)\right]\ \left[1+(n-1)\gamma\right]\ \frac{(a)_{n-1}}{(c)_{n-1}}|a_n|\le 1-\alpha \tag{2.1}$$

$-1\le\alpha<1, \gamma\ge 0$

Proof. It suffices to show that

$$\left|\frac{z\left(L(a,c;\gamma)f(z)\right)'}{L(a,c;\gamma)f(z)}-1\right|-Re\left\{\frac{z\left(\mathrm{L}(\mathrm{a,c};\gamma)f(z)\right)'}{L(a,c;\gamma)f(z)}-1\right\}\le 1-\alpha$$

We have

$$\left|\frac{z\left(L(a,c,\gamma)f(z)\right)'}{L(a,c,\gamma)f(z)}-1\right|-Re\left\{\frac{z\left(\mathrm{L}(\mathrm{a,c},\gamma)f(z)\right)'}{L(a,c,\gamma)f(z)}-1\right\}$$

$$\le 2\left|\frac{z\left(L(a,c,\gamma)f(z)\right)'}{L(a,c,\gamma)f(z)}-1\right|$$

$$\le\frac{2\sum_{n=2}^{\infty}(n-1)\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}{1-\sum_{n=2}^{\infty}\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{n=2}^{\infty}\left[2n-(\alpha+1)\right]\ \left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}|a_n|\le 1-\alpha$$

and this completes the proof.

Theorem 2.2. A necessary and sufficient condition for $f(z)$ of the form (1.2) to be in the class $TS(\alpha,\gamma)$, $-1\le\alpha<1$, $\gamma\ge 0$ is that

$$\sum_{n=2}^{\infty}\left[2n-(\alpha+1)\right]\ \left[1+(\mathrm{n}-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_n\le 1-\alpha \tag{2.2}$$

The result is sharp

Proof: In view of Theorem 2.1, we need only to prove the necessity If $f(z)\in TS(\alpha,\gamma)$ and z is real then

$$\frac{1-\sum_{n=2}^{\infty} n\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}}{1-\sum_{n=2}^{\infty}\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}}-\alpha \geq \left|\frac{\sum_{n=2}^{\infty}(n-1)\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}\right|$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty} 2n-(\alpha+1)\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_n \leq 1-\alpha$$

Corollary 2.1 If $f(z) \in TS(\alpha,\gamma)$ then

$$a_n \leq \frac{(1-\alpha)}{\left[2n-(\alpha+1)\right]\ \left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}} \text{ for } n \geq 2 \tag{2.3}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{\left[2n-(\alpha+1)\right]\ \left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}} z^n, n \geq 2 \tag{2.4}$$

If $\gamma = 0$ we get the following result of [1]

Corollary 2.2. If $f(z) \in TS(\alpha,\beta)$ then

$$a_n \leq \frac{(1-\alpha)}{\left[2n-(\alpha+1)\right]\frac{(a)_{n-1}}{(c)_{n-1}}}, n \geq 2 \tag{2.5}$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{\left[2n-(\alpha+1)\right]\frac{(a)_{n-1}}{(c)_{n-1}}} z^n,\ n \geq 2 \tag{2.6}$$

Theorem 2.3. Let $f(z)$ defined by (1.2) and $g(z)$ defined

$g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ be in the class $TS(\alpha,\gamma)$. Then the function $h(z)$ defined by

$$h(z) = (1-\lambda)f(z) + \lambda\, g(z) = z - \sum_{n=2}^{\infty} q_n z^n$$

where $q_n = (1-\lambda)a_n + \lambda\, b_n, 0 \leq \lambda < 1$ is also in the class $TS(\alpha,\gamma)$

Theorem 2.4. Let $f_1(z) = z$ and

$$f_n(z)=z-\frac{(1-\alpha)\ (c_{n-1})}{[2n-(\alpha+1)]\ \ [1+(n-1)\gamma](a)_{n-1}}z^n \tag{2.7}$$

For n = 2, 3, 4 .....

Then $f(z)\in TS(\alpha,\gamma)$ if and only it $f(z)$ can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n\ f_n(z)\,\text{where}\,\lambda_{\text{n}}\geq 0\,\text{and}\sum_{\text{n}=1}^{\infty}\lambda_{\text{n}}=1$$

The proof of the Theorem 2.4, follows on line similar to the proof of the theorem on extreme points given in silverman [9].

We prove the following theorem by defining $f_j(z)(j=1,2.......m)$ of the form

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n.j}z^n\ \text{ for }\ a_{n.j}\geq 0, z\in E \tag{2.8}$$

Theorem 2.5 Let the function $f_j(z)(j=1,2...m)$ defined by (2.8) be in the class $TS(\alpha_j,\gamma)$ $(j=1,2,.....m)$ respectively. Then the function $h(z)$ defined by

$$h(z)=z-\frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}a_{n.j}\right)z^n$$

is in the class $TS(\alpha,\gamma)$

where $\alpha=\begin{matrix}\min\\ 1\leq j\leq m\end{matrix}\{\alpha_j\}$ where $-1\leq\alpha_{\text{j}}<1$

Proof:

Since $f_j(z)\in TS(\alpha_j,\gamma),\ j=(1,2,3.....m)\ TS(\alpha_j,r)$ by applying theorem 2.2 to (2.8) we observe that

$$=\sum_{n=2}^{\infty}[2n-(\alpha+1)]\ \ [1+(n-1)\gamma]\frac{(a)_{n-1}}{(c)_{n-1}}\left(\frac{1}{m}\sum_{j=1}^{m}a_n,j\right)$$

$$=\frac{1}{m}\sum_{j=1}^{m}\left[\sum_{n=2}^{\infty}2n-(\alpha+1)\right]\ (1+(\text{n-1})\gamma)\ \frac{(\text{a})_{n-1}}{(c)_{n-1}}a_{n,j}$$

$$\leq\frac{1}{m}\sum_{j=1}^{\infty}(1-\alpha_j)$$

$$\leq(1-\alpha)$$

which in view of Theorem 2.2 again implies that $h(z)\in TS(\alpha,\gamma)$

Hence the theorem follows

Theorem 2.6 Let the function $f(z)$ defined by (1.2) be in the class $TS(\alpha,\gamma)$.

Then $f(z)$close to convex of order $\delta(0\leq\delta<1)$ in $|\text{z}|<r_1$ where

$$r_1 = \begin{matrix}\inf \\ n \geq 2\end{matrix} \left\{ \frac{(1-\delta)\left[2n-(\alpha+1)\right] \left[1+(n-1)\gamma\right] \frac{(a)_{n-1}}{(c)_{n-1}}}{n(1-\alpha)} \right\}^{1/n-1} \tag{2.9}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4)

Proof: We must show that

$$|f'(z)-1| \leq 1-\delta \text{ for } |z| < r_1 \tag{2.10}$$

where r_1, is given by (2.9), Indeed we have

$$|f'(z)-1| \leq \sum_{n=2}^{\infty} n\, a_n |z|^{n-1}$$

Thus

$$|f'(z)-1| \leq 1-\delta$$

$$\text{if } \sum_{n=2}^{\infty} \left(\frac{n}{1-\delta}\right) a_n |z|^{n-1} \leq 1 \tag{2.11}$$

Using the fact, that $f \in TS(\alpha,\gamma)$ if and only if

$$\frac{\sum_{n=2}^{\infty}\left[2n-(\alpha+1)\right] \left[1+(n-1)\gamma\right] \frac{(a)_{n-1}}{(c)_{n-1}}}{1-\alpha} a_n \leq 1$$

we can say (2.11) is true if

$$\frac{\left(\frac{n}{1-\delta}\right) |z|^{n-1} \leq \left[2n-(\alpha+1)\right] \left[1+(n-1)\gamma\right] \frac{(a)_{n-1}}{(c)_{n-1}}}{1-\alpha}$$

that is, if

$$|z| \leq \left\{ \frac{(1-\delta)\left[2n-(\alpha+1)\right] \left[1+(n-1)\gamma\right] \frac{(a)_{n-1}}{(c)_{n-1}}}{n(1-\alpha)} \right\}^{1/n-1} ,n \geq 2$$

This completes the proof of Theorem

Theorem 2.7: Let the function $f(z)$ defined by (1.2) be in the class $TS(\alpha,\gamma)$

Then $f(z)$ is strarlike of order $\delta\,(0 \leq \delta < 1)$ in $|z| < r_2$ where

$$r_2 = \begin{matrix}\inf \\ n \geq 2\end{matrix} \left\{ \frac{(1-\delta) \left[2\mathrm{n}-(\alpha+1)\right] \left[1+(\mathrm{n}\text{-}1)\gamma\right]}{(n-\delta)\ (1\text{-}\alpha)} \frac{(a)_{n-1}}{(c)_{n-1}} \right\}^{1/n-1}$$

The result is sharp with the extermal function $f(z)$ given by (2.4)

Proof: Given $f \in A$ and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\delta \tag{2.12}$$

For the left hand side of (2.12) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1}<1$$

Using the fact that $f \in TS(\alpha,\gamma)$ if and only

$$\sum_{n=2}^{\infty}\left[2n-(\alpha+1)\right]\left[1+(\text{n-1})\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}a_n \le 1$$

we can say (2.12) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{\left[2n-(\alpha+1)\right]\left[1+(n-1)\gamma\right]}{1-\alpha}\frac{(a)_{n-1}}{(c)_{n-1}}$$

or equivalently,

$$|z| \le \left\{\frac{(1-\delta)\left[2n-(\alpha+1)\right]\left[(1+n-1)\gamma\right]}{(n-\delta)\ (1\text{-}\alpha)}\frac{(a)_{n-1}}{(c)_{n-1}}\right\}^{1/n-1}$$

which yields the starliekness of the family

Using the fact that $f(z)$ is convex if and only if $z\ f'(z)$ is stalike, we get the following corollary.

Corollary 2.3: Let the function $f(z)$ defined by (1.2) be in the class $TS(\alpha,\gamma)$. Then $f(z)$ is convex of order $\delta(0 \le \delta < 1)$ in $|\text{z}| < r_3$ where

$$r_3 = \inf_{n \ge 2}\left\{\frac{(1-\delta)\left[2n-(\alpha+1)\right]\left[1+(n-1)\gamma\right]\ \frac{(a)_{n-1}}{(c)_{n-1}}}{n(n-\delta)\ (1-\alpha)}\right\}^{1/n-1}$$

The result is sharp with external function $f(z)$ given by (2.4).

3. Partial Sums

Following the earlier works by Silverman [9] and Silvia [10] on partial sums of analytic functions. We consider in this section partial sums of functions in this class $TS(\alpha,\gamma)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f_k'(z)$

IOP Conf. Series: Materials Science and Engineering **263** (2017) 042157 doi:10.1088/1757-899X/263/4/042157

Theorem 3.1: Let $f(z) \in TS(\alpha,\gamma)$ be giving by (1.1) and define the partial sums $f(z)$ and $f_k(z)$ by $f_1(z) = z$ and

$$f_k(z) = z + \sum_{n=2}^{k} a_n z^n, (k \in N/1) \tag{3.1}$$

Suppose also that

$$\sum_{n=2}^{\infty} d_n |a_n| \le 1 \tag{3.2}$$

Where $d_n = \left[\frac{2n-(\alpha+1)}{(1-\alpha)}\right]\left[1+(n-1)\gamma\right]\frac{(a)_{n-1}}{(c)_{n-1}}$

Then $f \in TS(\alpha,\gamma)$ Further more,

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{d_{k+1}}, z \in E, k \in N \tag{3.3}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{d_{k+1}}{1+d_{k+1}} \tag{3.4}$$

Proof: For the coefficients d_n given by (3.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1 \tag{3.5}$$

Therefore we have

$$\sum_{n=2}^{k} |a_n| + d_{k+1} \sum_{n=k+1}^{\infty} |a_n| \le \sum_{n=2}^{\infty} d_n |a_n| \le 1 \tag{3.6}$$

by using the hypothesis (3.2). By setting

$$g_1(z) = d_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{d_k+1}\right)\right\}$$

$$= 1 + \frac{d_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \tag{3.7}$$

and applying (3.6), we find that

$$\left|\frac{g_1(z)-1}{g_2(z)+1}\right| \le \frac{d_{k+1}\sum_{n=k+1}^{\infty}|a_n|}{2 - 2\sum_{n=2}^{\infty}|a_n| - d_{k+1}\sum_{n=k+1}^{\infty}|a_n|}$$

$$\le 1, z \in E \tag{3.8}$$

which ready yields the assertion (3.3) of Theorem 3.1. In order to see that

$$f(z)=z+\frac{z^{k+1}}{d_{k+1}} \tag{3.9}$$

gives sharp result, we observe that for $z=re^{i\pi/k}$ that

$\frac{f(z)}{f_k(z)}=1+\frac{z^k}{d_{k+1}}\to 1-\frac{1}{d_{k+1}}$ as $z\to 1^-$ Similarly, if we take

$$g_2(z)-(1+d_{k+1})\left\{\frac{f_k(z)}{f(z)}-\frac{d_{k+1}}{1+d_{k+1}}\right\}$$

$$=1-\frac{(1+d_{n+1})\sum_{n=k+1}^{\infty}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}a_nz^{n-1}} \tag{3.10}$$

And making use of (3.6) we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right|\le\frac{(1+d_{k+1})\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{k}|a_n|-(1-d_{k+1})\sum_{n=k+1}^{\infty}|a_n|} \tag{3.11}$$

which leads is immediately to the assertion (3.4) of **Theorem 3.1**

The bound in (3.4) is sharp for each $k\in N$ with the extermal function $f(z)$ given by (3.9). The proof of the Theorem 3.1. is thus complete.

Theorem 3.2: If $f(z)$ of the form (1.1) satisfies the condition (2.1) then

$$Re\left\{\frac{f'(z)}{f_k'(z)}\right\}\ge 1-\frac{k+1}{d_{k+1}} \tag{3.12}$$

Proof:
By setting

$$g(z)=d_{k+1}\left\{\frac{f'(z)}{f_k'(z)}\right\}\ge 1-\frac{k+1}{d_{k+1}}$$

$$=\frac{1+\frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty}na_nz^{n-1}+\sum_{n=2}^{\infty}na_nz^{n-1}}{1+\sum_{n=2}^{k}na_nz^{n-1}}$$

$$=\frac{1+\frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty}na_nz^{n-1}}{1+\sum_{n=2}^{k}na_nz^{n-1}}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n|}{2-2\sum_{n=2}^{k} n|a_n| - \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n|} \tag{3.13}$$

Now

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le 1$$

If

$$\sum_{n=2}^{k} n|a_n| + \frac{d_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \le 1 \tag{3.14}$$

Since the left hand side of (4.14) is bounded above by $\sum_{n=2}^{k} d_n|a_n|$ if

$$\sum_{n=2}^{k}(d_n - n)|a_n| + \sum_{n=k+1}^{\infty} d_n - \frac{d_{k+1}}{k+1} n|a_n| \ge 0 \tag{3.15}$$

and the proof is complete. The result is sharp for the extremal function $f(z) = z + \frac{z^{k+1}}{d_{k+1}}$

Theorem 3.3: If $f(z)$ of the form (1.1) satisfies the condition (2.1) then

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \ge \frac{d_{k+1}}{k+1+d_{k+1}} \tag{3.16}$$

Proof: By setting

$$g(z) = [(k+1)+d_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{d_{k+1}}{k+1+d_{k+1}}\right\}$$

$$= 1 - \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum_{n=2}^{k} na_n z^{n-1}}$$

and making use of (3.15), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \le \frac{\left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n|a_n|}{2-2\sum_{n=2}^{k} n|a_n| - \left(1+\frac{d_{k+1}}{k+1}\right)\sum_{n=k+1}^{\infty} n|a_n|} \le 1$$

which leads us immediately to the assertion of the Theorem 3.3

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