Generalized Dandelin's Theorem

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Abstract. The paper gives a geometric proof of the theorem which states that in case of the plane section of a second-order surface of rotation (quadrics of rotation, QR), such conics as an ellipse, a hyperbola or a parabola (types of conic sections) are formed. The theorem supplements the well-known Dandelin's theorem which gives the geometric proof only for a circular cone and applies the proof to all QR, namely an ellipsoid, a hyperboloid, a paraboloid and a cylinder. That's why the considered theorem is known as the generalized Dandelin's theorem (GDT).

The GDT proof is based on a relatively unknown generalized directrix definition (GDD) of conics. The work outlines the GDD proof for all types of conics as their necessary and sufficient condition. Based on the GDD, the author proves the GDT for all QR in case of a random position of the cutting plane. The graphical stereometric structures necessary for the proof are given. The implementation of the structures by 3d computer methods is considered. The article shows the examples of the builds made in the AutoCAD package.

The theorem is intended for the training course of theoretical training of elite student groups of architectural and construction specialties.

1. Introduction

The Dandelin's theorem, known since 1822, is a geometric proof of second-order curves' (conics: ellipse, hyperbola, parabola) formation when a circular cone is sectioned by a plane [1]. Along with the cone, Dandelin had adduced the proof for one-sheet hyperboloid of rotation [2,3]. The chapter about conic sections in descriptive geometry course is based on the Dandelin's theorem [4-6]. The proof for conics' formation when a cone is sectioned by a plane is given. Meanwhile, there is no proof of conic formation in plane sections for other quadrics of rotation (they are one-sheet and two-sheeted hyperboloids, oblong and oblate ellipsoids, paraboloid); in this case the formation is explained on the ground of analytic geometry. The reason is that the geometrical proof for the general case of quadrics of rotation is unknown.

Active practical implementation of computational 3D methods in geometric modelling makes a wide range of geometric models and problems accessible for the basic educational process [7-10]. Conse-quently, the generalized Dandelin's theorem also has arisen interest.

Statement of the Theorem: "flat section of a random quadric of rotation is a conic".

Our interest to the generalized Dandelin's theorem is additionally caused by the discovered opportunity of geometrically precise construction of mutually tangent conics [11,12]. It is experimentally shown, that the Dandelin spheres exist for all the quadrics of rotation. But it is noted, that the previous conclusion is drawn on the basis of plausible arguments and is not evidence [13,14].

The proof of generalized Dandelin's theorem provided in this article is based on researches [15, 16, 17] in which the possibility of geometrical proof of the generalized Dandelin's theorem on the basis of little-known generalized directrix definition of conics is shown.

The objective of this research paper is to provide geometric proof for the generalized Dandelin's theorem.

As in classic Dandelin's theorem, the considered generalized theorem sets aside the known special cases of flat sections of quadrics, which are concluded from the definitions of these quadrics.

The given geometric constructions are completed in AutoCAD package. All the constructions can be duplicated in any modern CAD graphic editor.
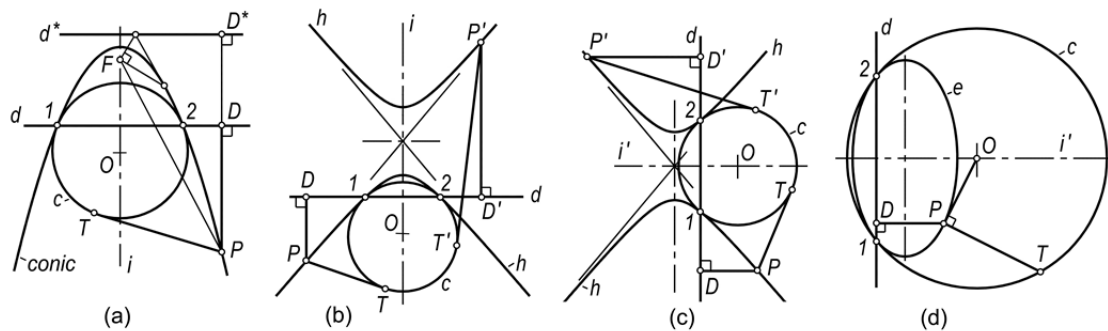

Figure 1. Variants of the generalized directrix definition.

2. Generalized directrix definition of conics

There is a generalized definition of conics as a geometrical locus with the constant parameter $\varepsilon = \dfrac{PF}{PD^*}$, where P is a random point of a conic, F is a focus point, d* is a directrix [18], see Figure 1(a). However, this definition does not lead to proving the generalized Dandelin's theorem.

In order to prove the generalized Dandelin's theorem, let us use another so-called "generalized directrix definition" of a conic [15-17]. Let us draw circumference c of random radius, tangent to conic *conic* in points 1,2, and then draw a so-called "directrix" – segment d (1,2). According to the generalized Dandelin's theorem, the ratio $q = \dfrac{PT}{PD} = \text{const}$, where PT is tangential distance from a random point P of conic *conic* till the circumference, and PD is the distance from point P till the "directrix". Thus, a conic is a geometrical locus with constant value of parameter q.

Value of $PT = \sqrt{|a|}$ where a is the valence of point P relative to circumference c [19]. For the point placed outside the circumference, PT is a segment of a tangent line, see Figure 1(a–c). For the point placed inside the circumference, PT is a half of the circumference's shortest chord drawn from point P, see Figure 1(d).

For a parabola, the center of tangent circumference is situated on its axis i, see Figure 1(a). For a hyperbola, the center of circumference can be situated on its transverse axis i, see Figure 1(b), or its conjugate axis i', see Figure 1(c). For an ellipse, it is either longer axis i, see Figure 1(a), or shorter axis i', see Figure 1(d).

Parameter constancy q can be experimentally verified in any CAD graphical package. To do this, construct a conic and a circle tangent to it. Then we make measurements for its several random points P. For example, see Figure 1(b, c), we check, that $= \dfrac{PT}{PD} = \dfrac{PT}{PD}$. Having constructed tangent circumferences of different radiuses, we can make sure that for the given conic, parameter q does not depend on the radius of this circumference.

3. Generalized directrix definition as a necessary condition of a conic

Let us consider the variants of generalized directrix definition (GDD) given in Figure 1(a,b). It is known that any conic *conic* $\subset \delta$, see Figure 2 (a), can be obtained as a section by the plane δ of any circular cone *Cone* [20, 21]. From the set of spheres inscribed in the cone, we find the sphere *S* whose section (the circle $c = S \cap \delta$) touches the conic at the points 1,2, see Figure 2 (b). The construction of such a sphere can be found in [22,23].
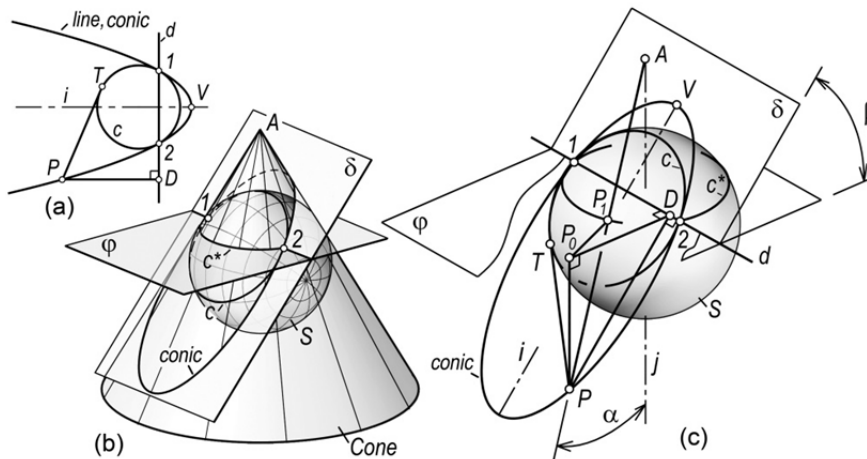

Figure 2. A proof of the generalized directrix definition.

We construct segment *d* (*1,2*), fix plane $\varphi \perp j$, $\varphi \supset d$, construct circumference $c^* = \varphi \cap S$ (or $c^* = \varphi \cap Cone$), see Figure 2(c). Then we set a random point of the conic $P \subset conic$. In plane δ we construct segment *PT* tangent to circumference *c* (*T* is the tangency point), and segment $PD \perp d$.

It is necessary to prove the stability of parameter $q = \dfrac{PT}{PD}$ for all the points of conic.

Construct $PP_0 \perp \varphi$ ($P_0 \subset \varphi$), the generatric of cone *AP*, and determine point $P_1 = AP \cap c^*$. Construct segment P_0P_1.

We should take into account that $PT = PP_1$ as segments of the tangent lines from a common point to a single sphere. Therefore:

$$q = \frac{PT}{PD} = \frac{PP_1}{PD} = \frac{PP_1}{PP_0} \cdot \frac{PP_0}{PD} = \frac{\sin \beta}{\cos \alpha} \qquad (1)$$

where α is the angle between axis *j* of the cone and its generatrices; β is the angle between planes δ and φ.

Considering that angles α and β are constant for all the points of the conic, we can conclude that parameter q does not depend on the position of point P, i.e. isn't the constant parameter for all the points of the conic.

We can also see that the value of q does not depend on radius of the circumference, inscribed in the conic of circumference c, because the parameters of this circumference are not included in the determined expression of q.

Since the cone Cone is given arbitrarily, the parameter q does not depend on the angle α.

4. Generalized directrix definition as a sufficient condition of a conic

<u>Given:</u> A curve *line,* for which the following conditions are met: $q = \dfrac{PT}{PD} = const$, see Figure 2(*a*), where points T, D are determined by constructing a circle tangent to curve. <u>Prove</u>, that line *line* is a conic, i.e. can be presented as a section of some cone.

Set a random point $P \subset line$ and determine the value of $q_1 = \dfrac{PT}{PD}$. Assume that this value is realized for all the points $P \subset line$.

Construct a cone *Cone* and an inscribed sphere *S,* see Figure 2(b). At that, *j* is their common axis, circumference *c* is a cross-section of sphere *S*, and point *P′* belongs to the surface of cone *Cone*. In order to create such construction, after insignificant adjustment we can use a parametric model [23]. Let us demonstrate that all points of line *line* belong to this cone.

Indicate $\varphi \perp j$, $\varphi \supset d$ *(1,2)*. Construct the following segments: *PT* tangent to circumference *c*, $PD \perp d$, $PP_0 \perp \varphi$ ($P_0 \subset \varphi$) and $P_0 D \perp d$, see Figure 2(c).

From point *P* draw a tangent line *PA* to sphere *S,* which crosses the axis *j*. Determine point $P_1 = PA \cap \varphi$. Considering that $PP_1 = PT,$ as segments tangent to sphere *S* from the common point *P,* and repeating (1), we obtain:

$$q = \frac{\sin \beta}{\cos \alpha} = q_1 = const$$

As long as *sin* β and parameter q_1 are constant values for all the points of curve *line*, the value of *cos* α and angle α are also constant values.

Therefore, all the tangent lines drawn to sphere *S* from the points of curve *line* and crossing the axis *j*, form a constant angle with this axis. This only becomes possible if the tangent lines are generatrices of a single cone. Thus, all the points belong to the surface of the cone, and curve *line* is a conic.

It is known, that Eccentricity of a conic is determined as follows [18, 21] $\varepsilon = \dfrac{\sin \beta}{\cos \alpha}$, where angles α and β are the same with the angles when deriving the generalized directrix definition, see Figure 2(c). Hence, *q* as a parameter of the generalized directrix definition and the Eccentricity of a conic are equal, i.e $q = \varepsilon$. We should emphasize, that this conclusion is only related to the considered variants of the generalized directrix definition, see Figure 1(a, b).

5. A proof of the generalized Dandelin's theorem
The received proof of the generalized directrix definition allows turning to a proof of the generalized Dandelin's theorem.
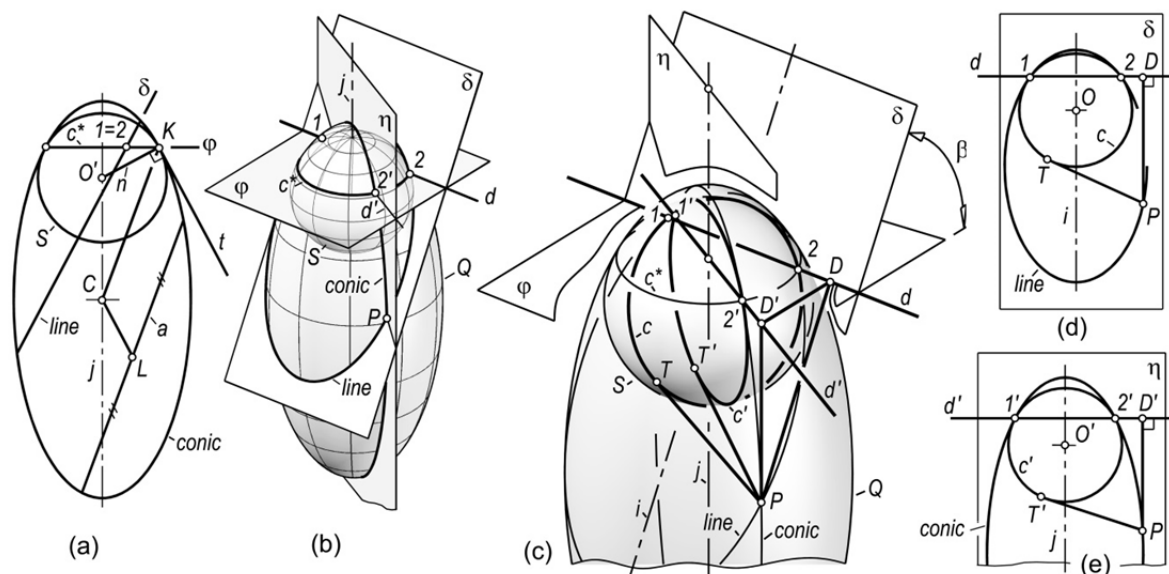

Figure 3. A proof of the generalized theorem.

<u>Given</u>: a quadric of rotation Q and a sectional plane δ. The quadric is formed by rotation of a conic *conic* around axis j. It is necessary to prove that the section curve *line* is a conic, see Figure 3(a, b).

Let us consider an example, in which, for illustrative purposes, quadric Q is an oblong ellipsoid. Introduce a sectional plane $\varphi \perp j$. Set the position of plane φ, so that the right line $d = \varphi \cap \delta$ would cross quadric Q in points 1 and 2. Then we construct a circumference $c^* = \varphi \cap Q$.

Inscribe sphere S into the quadric. For the sphere, circumference c^* is a tangency line of the sphere and the quadric. In order to determine the central point O' and the radius of this sphere, we should perform constructions by definition of a tangent line to a conic in the given point [24], see Figure 3(a): using the chord method, determine central point C of the conic *conic*, construct chord $a \parallel CK$; tangent line $t \parallel CL$; normal line $n \perp t$. We receive point $O' = n \cap j$ which is the center of sphere S. The parameterization can be applied [22].

In plane δ construct circumference $c = \delta \cap S$, see Figure 3(c, d). Since sphere S is inscribed in quadric Q, circumference c is inscribed in line *line* so they have two tangent points – points *1* and *2*. Construct segment d (*1,2*). Choose a random point $P \subset$ *line* and construct segment $PD \perp d$ and segment PT tangent to circumference c.

Introduce additional plane η (j, P), see Figure 3(b). Construct a cross-section of quadric Q and sphere S by this plane, see Figure 3(e). Crossing with the quadric occurs along conic *conic*, and with the sphere – along circumference c' (its meridian). Circumference c' and conic *conic* have tangency points *1′ and 2′*. In the same plane η we construct segments d'(*1′,2′*), $PD' \perp d'$, and segment PT' tangent to circumference c'. According to the generalized directrix definition as a necessary condition of conic *conic*, we receive $\dfrac{PT}{PD} = q = const$.

Taking into account that $PT = PT'$ as segments of the tangency lines to sphere S from common point P, then for the curve *line*:

$$q = \frac{PT}{PD} = \frac{PT}{PD} = \frac{PT}{PD} \cdot \frac{PD}{PD} = q \cdot \frac{PD}{PD}$$

Since $\eta \perp \varphi$, so $PD' \perp DD'$ and the relation $PD'/PD = sin\ \beta$, where β is the angle between plane δ and plane φ; the angle is constant for all the points of line *line*. Therefore:

$$q = q' \cdot sin\beta = const$$

That means, according to the generalized directrix definition as a sufficient condition of a conic, the *line* is a conic.

By moving plane φ along axis j, we receive a set of spheres S inscribed in quadric Q. Since angle β retain its value, the conclusion made about a conic's formation does not depend on radius of the sphere. The theorem is proved.

The given algorithm of the proof is correct for all quadrics of rotation. Depending on the type of a quadric and the position of plane δ, considered above different variants of the generalized directrix definition can be a part of the proof, see Figure 1.

6. Conclusion
The proof of the theorem is considered on the example of one quadric - an elongated ellipsoid. For the remaining quadrics the proof is given in [23].

A proof of the generalized Dandelin's theorem is based on the generalized directrix definition of conics. Despite the great amount of research papers on the properties of conics (the works of Menaechmus, Apollonius and the modern works on conics construction [11, 22 et al.]), the generalized directrix definition of conics remains an open question. For that matter, a proof of the generalized directrix definition for all types of conics is proposed in this article [23].

The proof of the generalized directrix definition is performed for conics as for specified cross-sections of a right circular cone. In our opinion, this allows to eliminate logical contradiction of works [15, 16, 17], in which the proof of the generalized directrix definition is performed on the basis of cross-sections of one-sheet hyperboloid of rotation, although the presence of conics in cross-sections hasn't been proved yet.

Special attention is paid to proving the generalized directrix definition to be a necessary and sufficient property of conics. This is caused by the fact that in the process of proving the generalized Dandelin's theorem, a collation of two cross-sections is performed. In one of the cross-sections, the generalized directrix definition is applied as a necessary property of conics, in another one – as their sufficient property.

Parameter q, introduced as a conic's characteristic based on the generalized directrix definition, is analogous to the parameter ε. Discovered correlation between parameters q and ε allows determine the type of a conic on the basis of the value of parameter q (same is for parameter ε).

The given proof of generalized directrix definition and generalized Dandelin's theorem possess constructive and geometric character (opposed to analytical proof). This corresponds with the specificity of geometrical and graphic training of students, and gives grounds for applying the generalized Dandelin's theorem in the educational process of Graphic departments, both in lecture courses and as case studies of geometrically accurate 3D computer constructions.

Applied significance of the considered theorem is in the geometric nature of its proof. In contrast to the proofs known from analytic geometry, given proofs of the generalized directrix definition and generalized Dandelin's theorem are visual; they are related to the school stereometric training of students and correspond to the specific geometrical-graphic training of students majoring in engineering. This gives us grounds for applying the generalized Dandelin's theorem in the educational process of Graphic departments, both in lecture courses and as case studies of geometrically accurate 3D computer constructions.

Acknowledgment

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