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Steady-state response of systems with fractional dampers

R Lewandowski[1] **and A Lenkowska**[1]

[1]Poznan University of Technology, 60-965 Poznań, ul. Piotrowo 5, Poland

Email: roman.lewandowski@put.poznan.pl

Abstract. In this paper, steady state vibrations of systems with built-up viscoelastic dampers are considered. The dampers are modeled using the fractional-derivative rheological models. The Caputo type fractional derivative definition is used. In particular, the steady state vibrations of systems are analysed. The solution to the steady state vibrations is written using real quantities. The effects induced by changes of environmental temperature are also considered and, in this context, the time-temperature superposition principle is adopted. The results of several parametric studies are also described and discussed in detail.

1. Introduction

Modern structures are higher, lighter and more flexible, constructed with the use of materials of higher strength, and optimally designed. However, these systems are more susceptible to dynamic loading and, in consequence, the amplitudes of vibration of such systems are sometimes too large; this can make it impossible to correctly utilize the structure or, in some cases, it can destroy the structures. In such cases, the structure's vibrations must be reduced. Many damper types are successfully developed to obtain a significant reduction of amplitudes of excessive vibration [1]. Viscoelastic (VE) dampers are a very promising class of dampers, to name just one. To describe the dynamic behavior of such dampers, a number of rheological models, characterized by the fractional derivative, can be used.

Steady state vibrations of a vibrating system with one degree of freedom and damping described by fractional derivatives were examined in paper [2]. Steady state vibrations caused by deterministic, harmonically changing forces were considered. In paper [3], the steady state vibrations of a linear and a non-linear system with one degree of freedom are analyzed. The damping of a system is described by means of a fractional derivative. In paper [4], non-linear steady state vibration of arches made by viscoelastic material are analyzed. The authors are using the residue harmonic homotopy method. Paper [5] is concerned with the steady state vibrations of a two-member plane truss system. Viscoelastic properties are described by the fractional Kelvin-Voigt model. The effects of fractional order and material modulus ratio on the system's responses are studied. An analysis of steady state vibrations of beams and frames with dampers, characterized by the fractional derivative, is conducted in paper [6].

In the present paper, steady state vibrations of systems with and without VE dampers are analyzed. To describe the dynamic behavior of dampers, the rheological model shown in Figure 1 is used. The equations describing the model's behavior contain the fractional derivatives. The model presented in Figure 1 is general because it contains a number of simpler models (both classic and fractional) as special cases, which are often used in the dynamic analysis of systems with the dampers (see [1, 6, 7]).

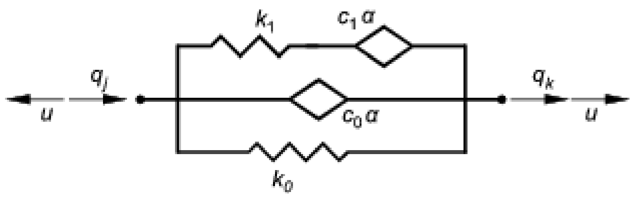

Figure 1. Diagram of rheological model of dampers.

2. Description of viscoelastic dampers

2.1. Description of spring-pot element

Different types of rheological models are used for describing the vibrations of dampers. Such models consist of viscous, elastic and spring-pot elements, connected in different ways. The most often used models are the viscous, Kelvin and Maxwell models. A combination of them provides more complex models which enable a more precise description of dampers. In models with a number of elements, the number of parameters significantly increases. This results in much more complex equations of motion. To eliminate these disadvantages, the fractional models of dampers are used to describe them. This enables a reduction of the number of damper parameters [7]. When fractional models are used, a precise description of damper's behavior with fewer parameters and wider frequency of excitation range is possible. Such models were proposed by Bagley and Torvik in [8]; they were used to describe the dynamic behavior of frames with dampers and sandwich beams, for example, in [9,10].

In this paper, the Caputo's definition of fractional derivative is used:

$$\mathrm{D}_t^{\alpha} q(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathrm{d}q(\bar{t})/\mathrm{d}\bar{t}}{(t-\bar{t})^{\alpha}} \mathrm{d}\bar{t} \quad , \tag{1}$$

where $\Gamma(\cdot)$ is a gamma function, $0 < \alpha < 1$ is an order of fractional derivative. Moreover, it is assumed that the lower limit of the integral in definition (1) was moved to $-\infty$.

This derivative is used to describe the behavior of the spring-pot element as proposed by Scott-Blair in paper [11]. In Figure 1, the elements are shown as the diamonds. The behavior of the spring-pot element is characterized by Eqn (2):

$$u(t) = c\, \mathrm{D}_t^{\alpha} \Delta q(t) \tag{2}$$

where $u(t)$ is force in the element, $\Delta q(t)$ is the difference of displacements of the element's ends, and c is the constant of the model. The coefficient c has an anomalous dimension $[\mathrm{Ns}^{\alpha}/m]$. The Scott-Blair element is one of which the properties are intermediate between those of elastic and viscous elements. For $\alpha = 0$, the element behaves like a spring and for $\alpha = 1$ it does like a viscous damper.

2.2. Steady state solution for dampers

Let us consider a fractional model of the damper shown in Figure 1. It is a general model, consisting of fractional Kelvin and Maxwell elements joined in parallel. In special cases, the model can turn into one of many simpler models of great practical importance, e.g., Scott-Blair, Kelvin, Maxwell or Zener model, in either fractional or classical versions (when $\alpha = 1$). The force in the analyzed model is a sum of forces in both constituent elements:

$$u(t) = u_0(t) + u_1(t)\,, \tag{3}$$

where index 0 denotes force in the Kelvin element, and index 1 is force in the Maxwell element.

The behavior of both elements is described by the following equations:

$$u_0(t) = k_0(1 + \tau_0^{\alpha} \mathrm{D}_t^{\alpha})(q_k - q_j)\,, \qquad u_1(t) + \tau_1^{\alpha} \mathrm{D}_t^{\alpha} u_1(t) = k_1 \tau_1^{\alpha} \mathrm{D}_t^{\alpha}(q_k - q_j)\ , \tag{4}$$

where $\tau_0^\alpha = c_0 / k_0$, $\tau_1^\alpha = c_1 / k_1$, symbols k_0 and k_1 denote the stiffness of the Kelvin and Maxwell elements, respectively, whereas q_j and q_k denote displacements of the ends of the damper's model.

If the damper vibrations are steady state vibrations, the following relationships are valid:

$$u(t) = u_c \cos\lambda t + u_s \sin\lambda t \ , \qquad u_0(t) = u_{0c} \cos\lambda t + u_{0s} \sin\lambda t \ , \tag{5}$$

$$u_1(t) = u_{1c} \cos\lambda t + u_{1s} \sin\lambda t \ , \qquad q_i(t) = q_{ic} \cos\lambda t + q_{is} \sin\lambda t \ . \tag{6}$$

Taking into account that

$$\mathrm{D}_t^\alpha \cos\lambda t = \lambda^\alpha \cos(\lambda t + \alpha\pi/2) \ , \quad \mathrm{D}_t^\alpha \sin\lambda t = \lambda^\alpha \sin(\lambda t + \alpha\pi/2) \ , \tag{7}$$

and, after substituting relationships (6) into Eqns (3) and (4), the following is obtained:

$$u_{0c}(t) = k_0[\chi_1(q_{kc} - q_{jc}) + \chi_2(q_{ks} - q_{js})] \ , \quad u_{0s}(t) = k_0[-\chi_2(q_{kc} - q_{jc}) + \chi_1(q_{ks} - q_{js})] \ , \tag{8}$$

$$u_{1c}(t) = k_1\Theta_1(q_{kc} - q_{jc}) + k_1\Theta_2(q_{ks} - q_{js}) \ , \quad u_{1c}(t) = -k_1\Theta_2(q_{kc} - q_{jc}) + k_1\Theta_1(q_{ks} - q_{js}) \ , \tag{9}$$

where symbols χ_1, χ_2, Θ_1 and Θ_2 are defined as:

$$\chi_1 = 1 + (\tau_0\lambda)^\alpha \cos\frac{\alpha\pi}{2} \ , \qquad \chi_2 = (\tau_0\lambda)^\alpha \sin\frac{\alpha\pi}{2} \ , \tag{10}$$

$$\Theta_1 = \frac{(\tau_1\lambda)^\alpha\left[(\tau_1\lambda)^\alpha + \cos\dfrac{\alpha\pi}{2}\right]}{1 + 2(\tau_1\lambda)^\alpha \cos\dfrac{\alpha\pi}{2} + (\tau_1\lambda)^{2\alpha}} \ , \qquad \Theta_2 = \frac{(\tau_1\lambda)^\alpha \sin\dfrac{\alpha\pi}{2}}{1 + 2(\tau_1\lambda)^\alpha \cos\dfrac{\alpha\pi}{2} + (\tau_1\lambda)^{2\alpha}} \ . \tag{11}$$

After substituting relationships (8) and (9) into Eqn (3), the following is obtained:

$$u_c = (k_0\chi_1 + k_1\Theta_1)(q_{kc} - q_{jc}) + (k_0\chi_2 + k_1\Theta_2)(q_{ks} - q_{js}) \ , \tag{12}$$

$$u_s = -(k_0\chi_2 + k_1\Theta_2)(q_{kc} - q_{jc}) + (k_0\chi_1 + k_1\Theta_1)(q_{ks} - q_{js}) \ . \tag{13}$$

2.3. Influence of temperature on damper's parameters

In order to determine the VE damper's response to changes of temperature, the time-temperature superposition principle can be used, as given by the following relationship:

$$K(t,T) = K(\tilde{\alpha}_T t_0, T_0) \, , \tag{14}$$

where K is called the complex modulus, t_0 and T_0 are the reference time and reference temperature, respectively. The symbol $\tilde{\alpha}_T$ denotes the so-called shift factor. The time-temperature superposition principle can be applied to the frequency domain and is sometimes named as the frequency-temperature correspondence principle [12]:

$$K(\lambda,T) = K(\tilde{\alpha}_T \lambda_0, T_0) \, , \tag{15}$$

where λ_0 is the reference frequency. In this paper only the horizontal shift factor is taken into account as the existing literature implies that for viscoelastic materials used in dampers the vertical shift factor seems to be equal to one (see [12]). However, this point needs deeper theoretical and experimental study.

The shift factor is calculated from some empirical formula. The following William-Landel-Ferry formula is often used:

$$\log \tilde{\alpha}_T = -C_1 \Delta T / (C_2 + \Delta T) \, , \tag{16}$$

where C_1 and C_2 are constants and $\Delta T = T - T_0$. Another formulae can also be used if they are more appropriate.

It is well known (see [7]) that the steady state vibration of dampers can be described using the complex modulus. The solution is in the following form:

$$\overline{u}(\lambda) = (K'(\lambda) + \mathrm{i}K''(\lambda))(\overline{q}_k - \overline{q}_j) \ , \tag{17}$$

where $K'(\lambda)$ is the storage modulus, $K''(\lambda)$ is the loss modulus and here $\mathrm{i} = \sqrt{-1}$.

The storage and loss moduli can be written as follows (see [7]):

$$K'(\lambda) = k_0\chi_1 + k_1\Theta_1 = k_0\left[1 + (\tau_0\lambda)^\alpha \cos(\alpha\pi/2)\right] + k_1 \frac{(\tau_1\lambda)^\alpha \left[(\tau_1\lambda)^\alpha + \cos(\alpha\pi/2)\right]}{1 + 2(\tau_1\lambda)^\alpha \cos(\alpha\pi/2) + (\tau_1\lambda)^{2\alpha}} \ , \tag{18}$$

$$K''(\lambda) = k_0\chi_2 + k_1\Theta_2 = k_0(\tau_0\lambda)^\alpha \sin(\alpha\pi/2) + k_1 \frac{(\tau_1\lambda)^\alpha \sin(\alpha\pi/2)}{1 + 2(\tau_1\lambda)^\alpha \cos(\alpha\pi/2) + (\tau_1\lambda)^{2\alpha}} \ . \tag{19}$$

Assuming that parameters k_0, k_1, τ_0 and τ_1 are related to the reference temperature T_0 and the reference frequency λ_0, Eqn (17) takes the form:

$$K'(\lambda_0, T_0) = k_0\left[1 + (\tau_0\lambda_0)^\alpha \cos(\alpha\pi/2)\right] + k_1 \frac{(\tau_1\lambda_0)^\alpha \left[(\tau_1\lambda_0)^\alpha + \cos(\alpha\pi/2)\right]}{1 + 2(\tau_1\lambda_0)^\alpha \cos(\alpha\pi/2) + (\tau_1\lambda_0)^{2\alpha}} \ . \tag{20}$$

When temperature is different from the reference value, the storage modulus can be described by:

$$K'(\lambda, T) = \tilde{k}_0\left[1 + (\tilde{\tau}_0\lambda)^\alpha \cos(\alpha\pi/2)\right] + \tilde{k}_1 \frac{(\tilde{\tau}_1\lambda)^\alpha \left[(\tilde{\tau}_1\lambda)^\alpha + \cos(\alpha\pi/2)\right]}{1 + 2(\tilde{\tau}_1\lambda)^\alpha \cos(\alpha\pi/2) + (\tilde{\tau}_1\lambda)^{2\alpha}} \ . \tag{21}$$

Taking into account the frequency-temperature correspondence principle, as expressed by Eqn (14), the above formula can be rewritten as follows:

$$K'(\lambda_0 = \tilde{\alpha}_T\lambda, T_0) = \tilde{k}_0\left[1 + (\tilde{\tau}_0\tilde{\alpha}_T\lambda)^\alpha \cos(\alpha\pi/2)\right] + k_1 \frac{(\tilde{\tau}_1\tilde{\alpha}_T\lambda)^\alpha \left[(\tilde{\tau}_1\tilde{\alpha}_T\lambda)^\alpha + \cos(\alpha\pi/2)\right]}{1 + 2(\tilde{\tau}_1\tilde{\alpha}_T\lambda)^\alpha \cos(\alpha\pi/2) + (\tilde{\tau}_1\tilde{\alpha}_T\lambda)^{2\alpha}} \ . \tag{22}$$

A comparison of Eqns (22) and (20) leads to the following results:

$$\tilde{k}_0 = k_0 \ , \qquad \tilde{k}_1 = k_1 \ , \qquad \tilde{\tau}_0 = \tilde{\alpha}_T\tau_0 \ , \qquad \tilde{\tau}_1 = \tilde{\alpha}_T\tau_1 \tag{23}$$

From Eqn (23), it can be established that only the parameters c_0 and c_1 of the damper model will change with changes of temperature according to the relationship:

$$\tilde{c}_0 = \alpha_T c_0 \ , \qquad \tilde{c}_1 = \alpha_T c_1 \ , \tag{24}$$

where $\alpha_T = \tilde{\alpha}_T^\alpha$ is the reduced shift factor.

3. Steady state vibration of systems with dampers

The motion of the system with dampers is described by the following equation [7]:

$$\mathbf{M}_k\ddot{\mathbf{q}}(t) + \mathbf{C}_k\dot{\mathbf{q}}(t) + \mathbf{K}_k\mathbf{q}(t) = \mathbf{P}(t) + \mathbf{F}(t) \tag{25}$$

In Eqn (25), the symbols $\mathbf{P}(t)$ and $\mathbf{F}(t)$ denote the vector of excitation forces and the vector of interacting dampers' forces, respectively, $\mathbf{M}_k, \mathbf{C}_k, \mathbf{K}_k$ means the matrix of mass, damping and stiffness, respectively. The vector of displacement of the system is denoted by $\mathbf{q}(t)$.

When the harmonic load described by Eqn (26) is applied:

$$\mathbf{P}(t) = \mathbf{P}_c \cos\lambda t + \mathbf{P}_s \sin\lambda t \ , \tag{26}$$

the dampers' interacting forces and steady state vibrations of the system can be described by the following equations:

$$\mathbf{F}(t) = \mathbf{F}_c \cos\lambda t + \mathbf{F}_s \sin\lambda t \ , \qquad \mathbf{q}(t) = \mathbf{q}_c \cos\lambda t + \mathbf{q}_s \sin\lambda t \ . \tag{27}$$

After introducing relationships (26) and (27) into Eqn (25), the equation of amplitude is obtained:

$$(\mathbf{K}_k - \lambda^2\mathbf{M}_k)\mathbf{q}_c + \lambda\mathbf{C}_k\mathbf{q}_s = \mathbf{P_c} + \mathbf{F_c} \ , \qquad -\lambda\mathbf{C}_k\mathbf{q}_c + (\mathbf{K}_k - \lambda^2\mathbf{M}_k)\mathbf{q}_s = \mathbf{P}_s + \mathbf{F}_s \ . \tag{28}$$

Moreover, the relationship between the vector $\mathbf{F}(t)$ and forces in dampers $u_i(t)$ ($i = 1,2,..,r$) are:

$$\mathbf{F}(t) = \sum_{i=1}^{r} \mathbf{e}_i u_i(t) \ , \tag{29}$$

where $\mathbf{e}_i$ is the allocation vector describing the position of the i-th damper on the structure.

Let us assume that the i-th damper is connected with the points k and j of which the displacements are q_k and q_j, respectively (see Figure 2). The following relationships can be written:

$$q_k(t) - q_j(t) = -\mathbf{e}_i^T\mathbf{q}(t) \ , \qquad q_{kc} - q_{jc} = -\mathbf{e}_i^T\mathbf{q}_c \ , \qquad q_{ks} - q_{js} = -\mathbf{e}_i^T\mathbf{q}_s \ . \tag{30}$$

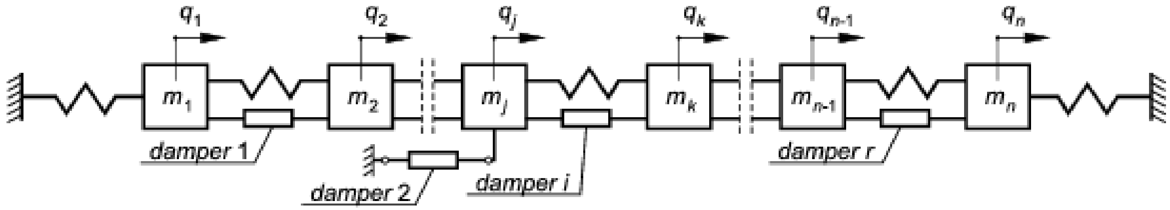

Figure 2. Diagram of a typical system with dampers.

Taking into account these principles, Eqns (12) and (13) can be rewritten as follows:

$$u_{ci} = -(k_{0i}\chi_{1i} + k_{1i}\Theta_{1i})\mathbf{e}_i^T\mathbf{q}_c - (k_{0i}\chi_{2i} + k_{1i}\Theta_{2i})\mathbf{e}_i^T\mathbf{q}_s \ , \tag{31}$$

$$u_{si} = -(-k_{0i}\chi_{2i} - k_{1i}\Theta_{2i})\mathbf{e}_i^T\mathbf{q}_c - (k_{0i}\chi_{1i} + k_{1i}\Theta_{1i})\mathbf{e}_i^T\mathbf{q}_s \ . \tag{32}$$

Index i means that parameters with the index concern the i-th dampers.

If the Eqns (31), (32) and (27) are substituted into Eqn (29), the vectors $\mathbf{F}_c$ and $\mathbf{F}_s$ can be expressed as:

$$\mathbf{F}_c = -(\mathbf{K}_{tv}^{(1)}\mathbf{q}_c + \mathbf{K}_{tv}^{(2)})\mathbf{q}_s \ , \qquad \mathbf{F}_s = -(-\mathbf{K}_{tv}^{(2)}\mathbf{q}_c + \mathbf{K}_{tv}^{(1)}\mathbf{q}_s) \ , \tag{33}$$

where:

$$\mathbf{K}_{tv}^{(1)} = \sum_{i=1}^{r}[k_{0i}\chi_{1i} + k_{1i}\Theta_{1i}]\,\mathbf{L}_i \ , \qquad \mathbf{K}_{tv}^{(2)} = \sum_{i=1}^{r}[k_{0i}\chi_{2i} + k_{1i}\Theta_{2i}]\,\mathbf{L}_i \ , \qquad \mathbf{L}_i = \mathbf{e}_i\mathbf{e}_i^T \tag{34}$$

After substituting Eqn (33) in the amplitude equation (28), it takes the following form:

$$(\mathbf{K}_k + \mathbf{K}_{tv}^{(1)} - \lambda^2\mathbf{M}_k)\mathbf{q}_c + (\lambda\mathbf{C}_k + \mathbf{K}_{tv}^{(2)})\mathbf{q}_s = \mathbf{P}_c \ , \tag{35}$$

$$-(\lambda\mathbf{C}_k + \mathbf{K}_{tv}^{(2)})\mathbf{q}_c + (\mathbf{K}_k + \mathbf{K}_{tv}^{(1)} - \lambda^2\mathbf{M}_k)\mathbf{q}_s = \mathbf{P}_s \ . \tag{36}$$

4. Balance of energy in systems with viscoelastic dampers

The balance of energy of the system in time t and $t+T$, where $T = 2\pi/\lambda$, can be formulated as follows:

$$E_k(t) + E_t(t) - E_w(t) = 0 \ , \qquad E_k(t+T) + E_t(t+T) - E_w(t+T) = 0 \ . \tag{37}$$

The symbols $E_k(t)$, $E_t(t)$ and $E_w(t)$ denote energy of the system, energy of the dampers, and work of the excitation forces in time t, respectively. The change of energy during one period of the cycle T is:

$$E_k(t+T) - E_k(t) + E_t(t+T) - E_t(t) - E_w(t+T) + E_w(t) = 0 \ . \tag{38}$$

The energy of the system consists of the kinetic, elastic and dissipated energies. The energy of the dampers consists of the kinetic and dissipated energies.

Because the system vibrates harmonically, changes of its kinetic and elastic energies are equal to zero. Then Eqn (38) can be rewritten in the form:

$$\Delta E_k + \Delta E_t = \Delta E_w \ , \tag{39}$$

where the symbols ΔE_k, ΔE_t and ΔE_w denote changes of the energy dissipated by the system, dissipated by the dampers and work of the external forces during one period of the cycle T, respectively. It can be assumed without losing the generality of considerations that $t = 0$.

Changes of both the elastic and dissipated energies of the *i*-th damper and vibrating harmonically with the period T are calculated as follows:

$$\Delta E_{ti} = \int_0^T u(t)\dot{x}(t)\mathrm{d}t \ , \tag{40}$$

where $x(t) = q_k - q_j$ is the difference of displacements of damper's ends (see Figure 1).

After substituting relationship (5) into (40) and integrating Eqn (40) with respect to time, the following is obtained:

$$\Delta E_{ti} = \pi(u_c x_s - u_s x_c) = \pi(k_0 \chi_2 + k_1 \Theta_2)(x_c^2 + x_s^2) \ , \tag{41}$$

where $x_s = q_{ks} - q_{js}$, $x_c = q_{kc} - q_{jc}$.

The damper vibrates harmonically, it means that the change of elastic energy is equal to zero in the damper and the relationships (40) and (41) are the definitions of change of the energy dissipated by the damper.

If the viscous damping forces are a load to the system, then a change of the energy dissipated by the system (without analyzing the dampers) is given by Eqn (42):

$$\Delta E_{tk} = \int_0^T \dot{\mathbf{q}}^T(t)\mathbf{C}_k\dot{\mathbf{q}}(t)\mathrm{d}t = \pi\lambda(\mathbf{q}_c^T\mathbf{C}_k\mathbf{q}_c + \mathbf{q}_s^T\mathbf{C}_k\mathbf{q}_s) \ , \tag{42}$$

where $\mathbf{C}_k$ is the matrix of viscous damping of the system.

A change of the work of the excitation forces can be calculated from the formula:

$$\Delta E_w = \int_0^T \mathbf{P}^T(t)\dot{\mathbf{q}}(t)\mathrm{d}t = \pi(\mathbf{P}_c^T\mathbf{q}_s - \mathbf{P}_s^T\mathbf{q}_c) \ . \tag{43}$$

Temperature in VE dampers can change with environmental temperature and during the process of energy dissipation when dissipated energy is changed to the heat. The latter is so-called the self-heating phenomenon. The increase of temperature is important when structures are subjected to winds. More detailed analysis of effects of temperature on VE dampers can be found in [12-17].

5. Results of numerical analysis

5.1. Example 1 – basic properties of systems with VE dampers

The system shown in Figure 3 was analyzed. A force $\mathbf{P}(t) = \mathbf{P}_c \cos\lambda t$ was applied upon it where $\mathbf{P}_c = [0.0,\ 0.0,\ 0.0,\ 5.0\,\mathrm{kN}]^\mathrm{T}$. It is assumed that: $m_1 = m_2 = m_3 = 44000.0\,\mathrm{kg}$, $m_4 = 22000.0\,\mathrm{kg}$, $k_1 = k_2 = k_3 = 150.0\,\mathrm{MN/m}$, $k_4 = 45.0\,\mathrm{MN/m}$. The damping matrix was determined based on the equation: $\mathbf{C}_k = \alpha_0\mathbf{M}_k + \alpha_1\mathbf{K}_k$ where $\alpha_0 = 0.34$ and $\alpha_1 = 0.000533$. In all examples, dampers parameters used are similar to parameters of VE materials investigated in the paper [18].

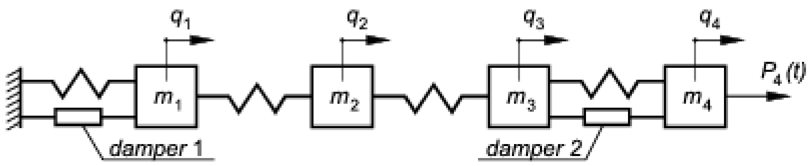

Figure 3. Diagram of the analysed system.

Systems with and without viscoelastic dampers, arranged as illustrated in Figure 3, were analyzed. The dampers were characterized by the following parameters: $\alpha = 0.7$, $c_{t0} = 0.2\,\mathrm{MNs}^{\alpha}/\mathrm{m}$, $k_{t0} = 150.0\,\mathrm{MN/m}$, $c_{t1} = 0{,}03\mathrm{MNs}^{\alpha}/\mathrm{m}$, $k_{t1} = 4.7\,\mathrm{MN/m}$.

The response curves were plotted for the systems without dampers and with different rheological damper models. The curves are shown in Figure 4. Moreover, the maximum amplitudes of resonance vibrations of a system with different types of dampers are shown in Table 1.

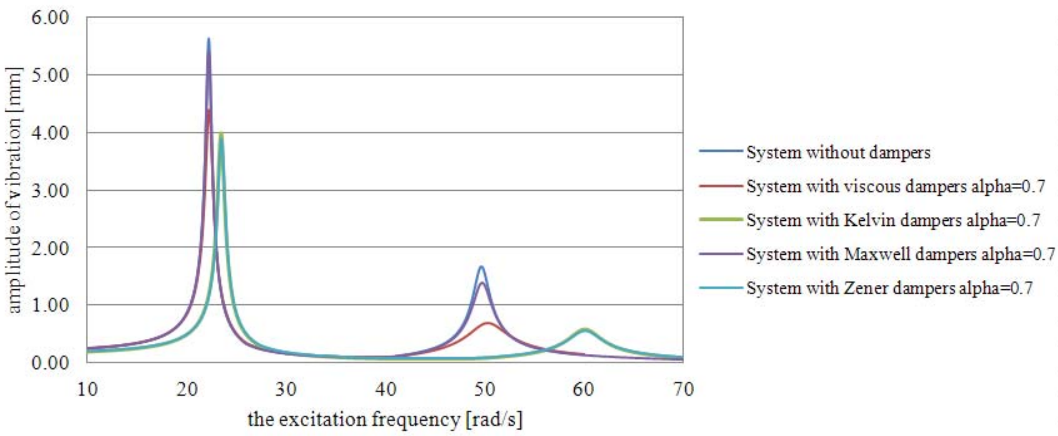

Figure 4. Response curves for system with dampers modeled by different rheological models.

Table 1. Amplitudes of vibration in resonances for system with dampers of different types.

Damper type	First resonance		Second resonance	
	Resonance frequency [rad/s]	Amplitude of vibration [m]	Resonance frequency [rad/s]	Amplitude of vibration [m]
no dampers	22.2	0.005638	49.7	0.001674
Spring pot	22.2	0.004383	50.3	0.000685
Kelvin	23.5	0.003993	60.1	0.000578
Maxwell	22.2	0.005418	49.8	0.001398
Zener	23.5	0.003908	60.1	0.000543

Figures 5 and 6 show the results of calculations of the energy dissipated by the dampers. The same results are presented in Table 2, where the maximum values of dissipated energies and work of external forces are given. Figure 5 shows how the dissipation energy changes with excitation frequency for different types of dampers.

Figure 6 shows changes of energy dissipated by individual Zener dampers vs. excitation frequency. The analogous graphs are similar as for other types of dampers. In Table 3, the maximum values of energy dissipated by the individual dampers, modeled as different rheological types are shown in both resonance regions.

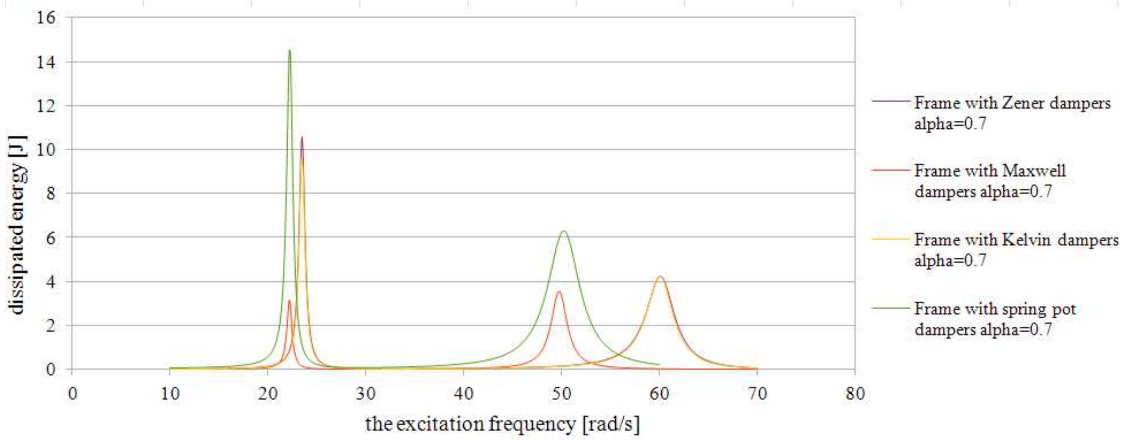

Figure 5. Energy dissipated by system with dampers vs. excitation frequency.

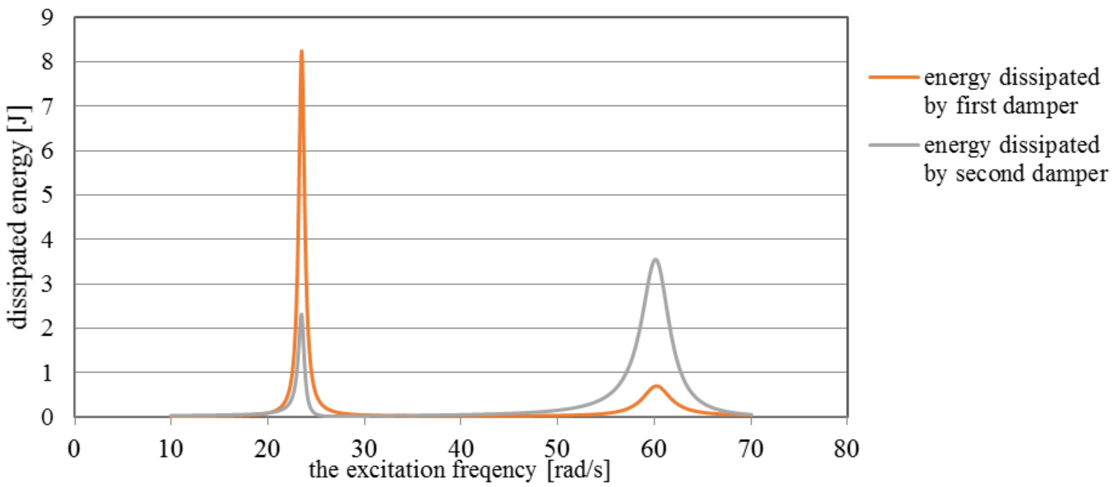

Figure 6. Energy dissipated by first and second dampers modeled by the Zener model.

Table 2. Maximum work of external force and maximum dissipation energy.

Damper type	1st resonance			2nd resonance		
	Work of external force [J]	Energy dissipated by system [J]	Energy dissipated by dampers [J]	Work of external force [J]	Energy dissipated by system [J]	Energy dissipated by dampers [J]
no damper	88.337	88.337	-	26.262	26.262	-
spring pot	68.423	53.970	14.502	10.747	4.464	6.285
Kelvin	62.720	53.065	9.654	9.076	4.845	4.232
Maxwell	84.766	81.591	3.175	21.925	18.343	3.581
Zener	61.357	50.811	10.546	8.521	4.293	4.236

Table 3. Maximum energy dissipated by single dampers in two resonance regions.

Damper type	Energy dissipated by dampers [J]			
	1st resonance		2nd resonance	
	1st damper	2nd damper	1st damper	2nd damper
Spring pot	9.153	5.490	0.451	5.834
Kelvin	7.543	2.110	0.685	3.550
Maxwell	1.968	1.206	0.247	3.333
Zener	8.240	2.306	0.690	3.548

Several remarks can be formulated on the basis of calculations conducted by the authors:

1. Amplitudes of resonance vibrations may be significantly reduced using the mentioned dampers.
2. The extent of reduction of vibrations depends on damper type. Dampers modeled with the spring-pot and Maxwell models reduce the amplitude of vibrations to a smaller degree, compared with dampers modeled with the Kelvin and Zener models.
3. The resonance frequencies of vibrations of the system with the spring pot and Maxwell model dampers do not change noticeably, compared with the systems with no dampers. In contrast, frequencies of resonance vibrations of systems with the Kelvin and Zener type of dampers may increase significantly. The difference depends on the proportion between the stiffness coefficients of the system and the stiffness coefficients of the dampers.
4. In the case discussed in this paper, the first damper dissipates energy mainly in the first resonance region and the second damper does in the second one, regardless of the damper model.

5.2. Example 2 – temperature effects on responses of systems with dampers

The system shown in Figure 3 was analyzed. For the purpose of calculations, the same set of data as in Example 1 is adopted. Dampers are described with the fractional Zener model. The following values of dampers' parameters were used (valid for the reference temperature $T_0 = 20\ ^{\circ}\mathrm{C}$): $\alpha = 0.7$, $k_0 = 542.85\ \mathrm{kN/m}$, $k_1 = 332.757\ \mathrm{MN/m}$, $c_{1,ref} = 521.84\ \mathrm{kNs}^{\alpha}/\mathrm{m}$ $c_{0,ref} = 0.0\ \mathrm{Ns}^{\alpha}/\mathrm{m}$, $C_1 = 9.23$ and $C_2 = 141.2$. Calculation were performed for the following range of temperature: $T_{\min} = -30^{o}C$, $T_{\max} = 30^{o}C$.

Figure 7 shows changes of the shift factor value vs. temperature. It can be noticed that $\tilde{\alpha}_T$ decreases with increasing temperature. Figure 8 illustrates the nature of changes of maximum vibrations' amplitudes in the first resonance region depending on damper's temperature. The maximum amplitudes decrease for the lower range of temperatures and they start to increase after exceeding $T = -20^{o}C$. The change of temperature causes significant changes in the vibrations' amplitude.

6. Concluding remarks

The steady state responses of systems with viscoelastic dampers are considered in the paper. A set of fractional rheological models are considered as models of dampers. The Caputo fractional derivatives are used to describe viscoelastic dampers. The impact of environmental temperature on the behavior of a system with viscoelastic dampers is considered using the time-temperature superposition principle.

Temperature was found to have a significant impact on maximum amplitudes in the first and second resonance regions. The unexpected behavior of the system with dampers was observed in the first resonance region.

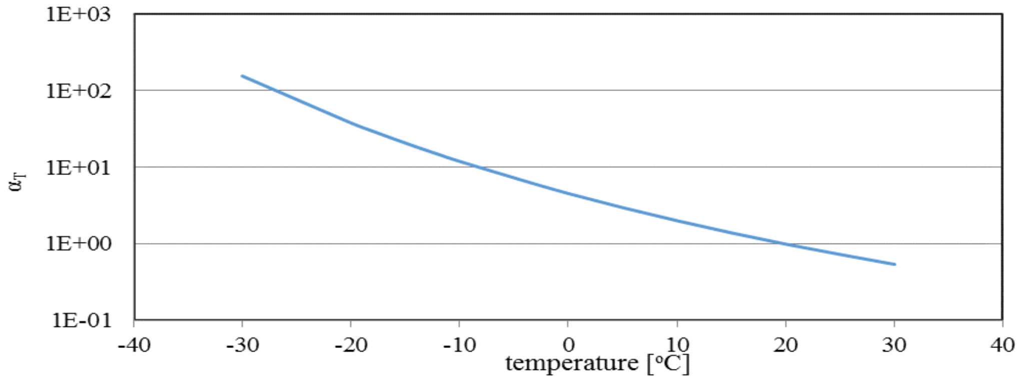

Figure 7. The shift factor $\tilde{\alpha}_T$ as a function of temperature of dampers.

Moreover, the equation which describes the balance of energy of the system with dampers was derived and some remarks concerning the dissipated energy were formulated. In particular, it was shown that the energy dissipated in the resonance regions by the chosen damper provided valuable information about the damper's effectivness and showed which mode of vibration was mainly damped. This energy could be a good indicator of the damper's optimal position. The typical response curves of the system with dampers are determined and compared to ilustrate the properties of systems and possible reduction of amplitudes.

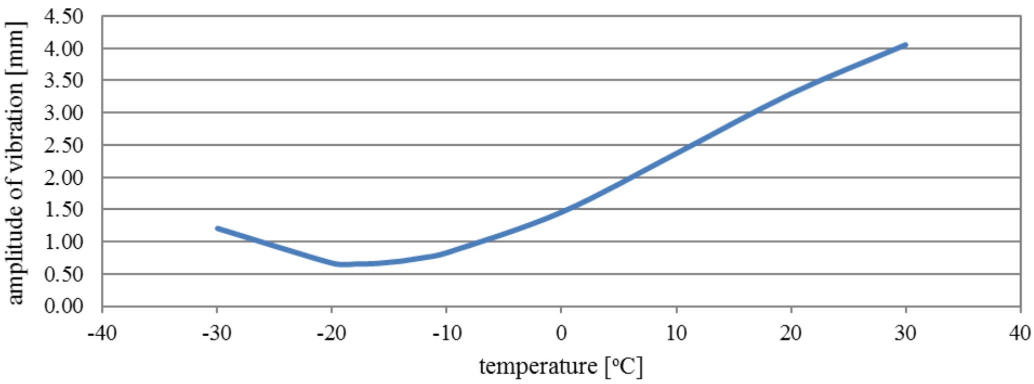

Figure 8. Maximum resonance amplitudes vs. temperature of dampers.

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