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Williams Element with Generalized Degrees of Freedom for Fracture Analysis of Multiple-Cracked Beam

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Abstract. In this paper, the method of finite element with generalized degrees of freedom (FEDOFs) is used to calculate the stress intensity factor (SIF) of multiple cracked beam and analysed the effect of minor cracks on the main crack SIF in different cases. Williams element is insensitive to the size of singular region. So that calculation efficiency is highly improved. Examples analysis validates that the SIF near the crack tip can be obtained directly though FEDOFs. And the result is well consistent with ANSYS solution and has a satisfied accuracy.

1. Introduction

Beam system is widely used in architectural engineering, bridge engineering and underground engineering. Structure is easily cracked cracks during application. And most of the cracks are multiple cracks. These cracks will reduce structural stiffness and integrity, which causes a serious threat on the structure. This paper is applied to fracture analysis of multiple cracked beam using FEDOFs compared with ANSYS.

The safety analysis of structure with cracks usually needs to determine the stress field or the displacement field near the crack tip. Due to SIF could be efficiently manifested as the degree of crack tip stress - strain field, crack tip SIF becomes an important research object on structural linear elastic fracture analysis. Researchers apply different kinds of methods to calculate the value of crack tip SIF on a single crack beam, such as mathematical and mechanical theory [1], the boundary collocation method [2], and the singularity element method [3]. However, the SIF study of multiple crack beam is rare no matter at home or abroad. Rohde [4] combined analytical theory with geometry factors proposed an efficient algorithm for calculating the crack tip SIF containing multiple boundary cracked beam. Ooi [5] integrated SBFEM(scaled boundary finite element method) with finite element, and used SBFEM in the extended area to analyze the structure displacement changes on three sided-cracks and double edge cracks beam under different loads and the cracks propagation with loading. Qingyuan Wang [6] applied the singularity element to analyze the three-point bending beam that contains three cracks. In additional, the influence of the change of minor crack length and its position modification to the main crack's crack tip SIF as well as mid-span deflection was discussed. Obviously, the research of multiple cracks fracture analysis still lack of achievement and remains to be furthered.

Like industry standards such as UML activity diagrams, Business Process Model and Notation and EPCs, Petri nets [7] offer a graphical notation for stepwise processes that include choice, iteration, and concurrent execution. Unlike these standards, Petri nets have an exact mathematical definition of their execution semantics, with a well-developed mathematical theory for process analysis. Williams generalized parameters element (W element) has been well applied [8-10] in fracture analysis that relates to the center-cracked plate and the three-point bending beam with higher precision and efficiency. This thesis will establish W unit calculating format for fracture analysis of multiple cracked beam. At the same time, using singularity element as contrast to verify the correctness of the results, and providing possible engineering advice.

2. Williams Element and SIF

2.1. Williams Element

Assuming *n* strips of vertical cracks are on the low edge of a simply supported beam, now taking 3 strips as example and discretizing them using finite element mesh, as shown in figure 1.

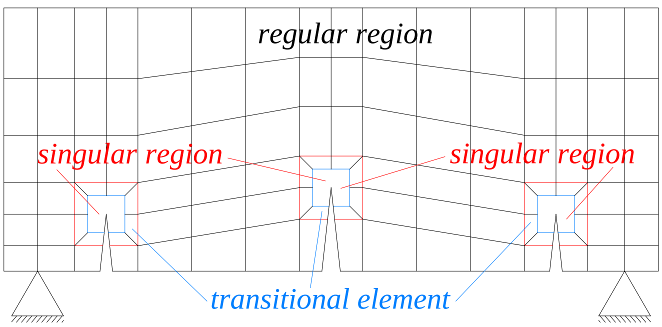

Figure 1. Discretized mesh with 3 edge cracks.

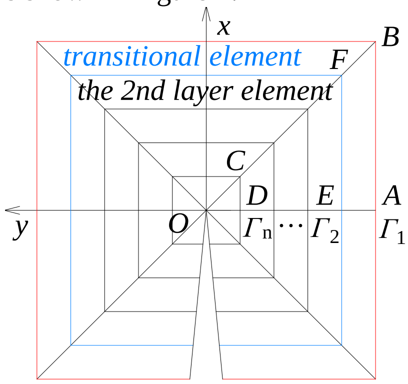

Figure 2. Discretized elements around the crack tip.

The interior of the red region represents the ranges of crack tip singular region, and out of it is regular region. It can be learned from reference [11], the size of the crack tip singular region almost has no effect on the result of SIF, and every sub-element crack tip region selects the same size of the singular region for easily to discretize finite element mesh. The outermost of the singular region called sub-element ABFE, own the same edge with adjacent regular region element, as a result, the element one part in the regular region, and another in the singular region, termed as transitional element, which is between the blue line and the red line in figure 1 and figure 2.Crack tip triangle element sub-element OCD is in the crack tip damaged region, when ignore its element stiffness contribution to the global stiffness, termed trapezoidal element as Williams element (W element), or Williams macro-element.

2.2. Sub-Element SIF at the Crack Tip

In this thesis, Williams series is selected as the arbitrary sub-element global displacement interpolation function. Thus its displacement field can be rewritten in matrix form, and intercept first *m* terms of the series:

$$\{w\}=[H(r,\theta)]\{a\} \tag{1}$$

where, $[H(r,\theta)]=\begin{bmatrix} 1 & 0 & r^{1/2}h_{11}^{x}(\theta) & r^{1/2}h_{12}^{x}(\theta) & \cdots & r^{m/2}h_{m1}^{x}(\theta) & r^{m/2}h_{m2}^{x}(\theta) \\ 0 & 1 & r^{1/2}h_{11}^{y}(\theta) & r^{1/2}h_{12}^{y}(\theta) & \cdots & r^{m/2}h_{m1}^{y}(\theta) & r^{m/2}h_{m2}^{y}(\theta) \end{bmatrix}$; $\{w\}=\{u,v\}^{\mathrm{T}}$;

$\{a\}=\{u_0 \quad v_0 \quad a_1 \quad b_1 \quad \cdots \quad a_i \quad b_i \quad \cdots \quad a_m \quad b_m\}^{\mathrm{T}}$. And has

$$h_{i1}^{x}(\theta)=\frac{1}{2G}[(\kappa+\frac{i}{2}+(-1)^{i})\cos\frac{i}{2}\theta-\frac{i}{2}\cos(\frac{i}{2}-2)\theta];h_{i2}^{x}(\theta)=\frac{-1}{2G}[(\kappa+\frac{i}{2}-(-1)^{i})\sin\frac{i}{2}\theta-\frac{i}{2}\sin(\frac{i}{2}-2)\theta]$$

$$h_{i1}^{y}(\theta)=\frac{1}{2G}[(\kappa-\frac{i}{2}-(-1)^{i})\sin\frac{i}{2}\theta+\frac{i}{2}\sin(\frac{i}{2}-2)\theta];h_{i2}^{y}(\theta)=\frac{1}{2G}[(\kappa-\frac{i}{2}+(-1)^{i})\cos\frac{i}{2}\theta+\frac{i}{2}\cos(\frac{i}{2}-2)\theta]$$

in which G is the shear modulus; κ is defined as: take 3-4μ in plane strain, take (3-μ)/(1+μ) in plane stress , μ is the Poisson ratio; a_i、b_i is the undetermined coefficients, namely the general parameters in equation (1), they can be defined from external loading and boundary condition; (r, θ) is the polar coordinate taking sub-element crack tip as origin.

Every stress components can be obtained according to Williams series displacement expression, and it can be known from the linear elastic fracture mechanics: SIF at the crack tip can be obtained from corresponding stress components. In order to avoid morbid equations, post-processing shear modulus G, so when $r\rightarrow 0$ and θ=0, one has:

$$K_{\mathrm{I}} = \sqrt{2\pi} G a_1 \ , \ K_{\mathrm{II}} = \sqrt{2\pi} G b_1 \tag{2}$$

Only needs obtaining corresponding column vector $\{a\}_i$ to each sub-element crack tip, and extracting general parameters a_1 and b_1 of each column vector, corresponding crack tip SIF can be obtained.

3. The Global Governing Equation

Solving global governing equation of domain. Discretizing the whole solution domain to n crack tip singular region and corresponding transition region and peripheral regular region, each crack tip singular region and transition region are discretized to multiple Williams element. It can learned from the constitution of Williams element that its element stiffness equation can be divided into blocks as follow:

$$\begin{bmatrix} [K]_{\mathrm{bb}}^{\mathrm{e}} & [K']_{\mathrm{bs}}^{\mathrm{e}} \\ [K']_{\mathrm{sb}}^{\mathrm{e}} & [K']_{\mathrm{ss}}^{\mathrm{e}} + [K'] \end{bmatrix} \begin{Bmatrix} \{w\}_{\mathrm{b}}^{\mathrm{e}} \\ \{a\} \end{Bmatrix} = \begin{Bmatrix} \{f\}_{\mathrm{b}}^{\mathrm{e}} \\ \{f\}_{\mathrm{s}}^{\mathrm{e}} + \{f'\} \end{Bmatrix} \tag{3}$$

where $[K']_{\mathrm{sb}}^{\mathrm{e}} = [T']^{\mathrm{T}} [K]_{\mathrm{sb}}^{\mathrm{e}} = [K']_{\mathrm{bs}}^{\mathrm{eT}}$; $[K']_{\mathrm{ss}}^{\mathrm{e}} = [T']^{\mathrm{T}} [K]_{\mathrm{ss}}^{\mathrm{e}} [T']$; $[K'] = \sum_{k=2}^{n} [K]^{(k)}$, in which b is the common boundary nodes of regular region and crack tip singular region , subscript s is the singular region.

Integrating the whole Williams element of n crack tip and the whole element stiffness of regular region to obtain global stiffness, and without considering the condition of loading in crack tip singular region, then solve global governing equation of domain:

$$\begin{bmatrix} [K]_{\mathrm{rr}} & [K]_{\mathrm{rb}} & [0] & [0] & [0] & [0] \\ [K]_{\mathrm{br}} & [K]_{\mathrm{bb}} & [K']_{\mathrm{bs_1}} & [K']_{\mathrm{bs_2}} & \cdots & [K']_{\mathrm{bs_n}} \\ [0] & [K']_{\mathrm{s_1b}} & [K']_{\mathrm{s_1s_1}} & [0] & [0] & [0] \\ [0] & [K']_{\mathrm{s_2b}} & [0] & [K']_{\mathrm{s_2s_2}} & [0] & [0] \\ [0] & \vdots & [0] & [0] & \ddots & [0] \\ [0] & [K']_{\mathrm{s_nb}} & [0] & [0] & [0] & [K']_{\mathrm{s_ns_n}} \end{bmatrix} \begin{Bmatrix} \{w\}_{\mathrm{r}} \\ \{w\}_{\mathrm{b}} \\ \{a\}_1 \\ \{a\}_2 \\ \vdots \\ \{a\}_{\mathrm{n}} \end{Bmatrix} = \begin{Bmatrix} \{f\}_{\mathrm{r}} \\ \{f\}_{\mathrm{b}} \\ \{0\} \\ \{0\} \\ \vdots \\ \{0\} \end{Bmatrix} \tag{4}$$

where subscript r is the regular region. According to the boundary conditions, solving equation (4) then generalized parameters $\{a\}_1$、$\{a\}_2$,…, $\{a\}_n$ of each crack tip region can be obtain, extracting general parameters a_1 and b_1 of each column vector, substituting it into equation (2) so SIF of each crack tip can be obtain.

4. Numerical Example

A main crack of length c_1 and two minor cracks of length c_2 exist in a three-point bending beam with span length l=4000 mm, section size $b\times h$=250 mm×600 mm, elastic modulus E=3.0×10^4 MPa and Poisson ratio μ=0.2. A concentrated load P=2.5 kN is concentrate in mid-span, as shown in figure 3. Calculating the changing law of main crack and two minor cracks SIF when the minor cracks' length and position have changed by Williams elements.

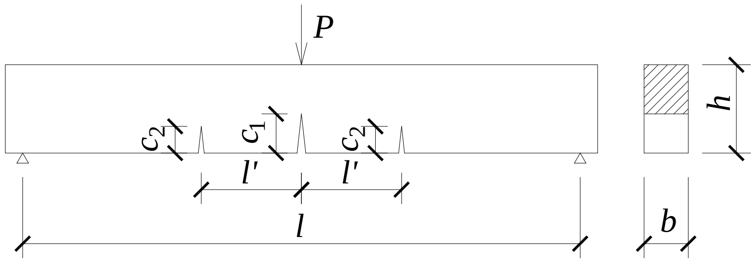

Figure 3. Three-point bending beam with three edge cracks.

4.1. Preserve Minor Crack Length Constant with Position Changed

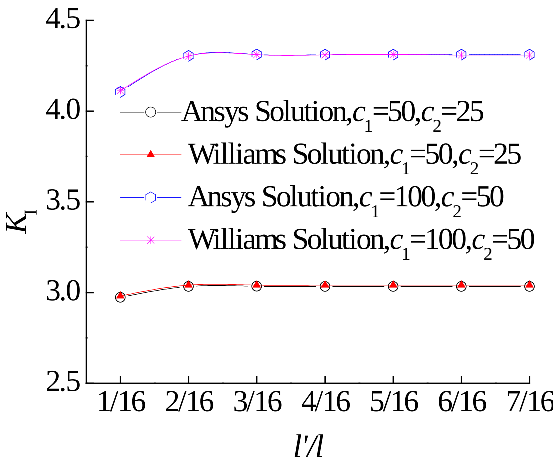

Figure 4. Main crack K_{I} versus minor cracks' position changed.

Figure 4 shows that when l'/l smaller than 1/8(that is near the mid-span), there is a decreasing trend of main crack K_{I}; when l'/l is larger than 1/8, the change of the position of the minor crack has no effect on the K_{I} of the main crack.

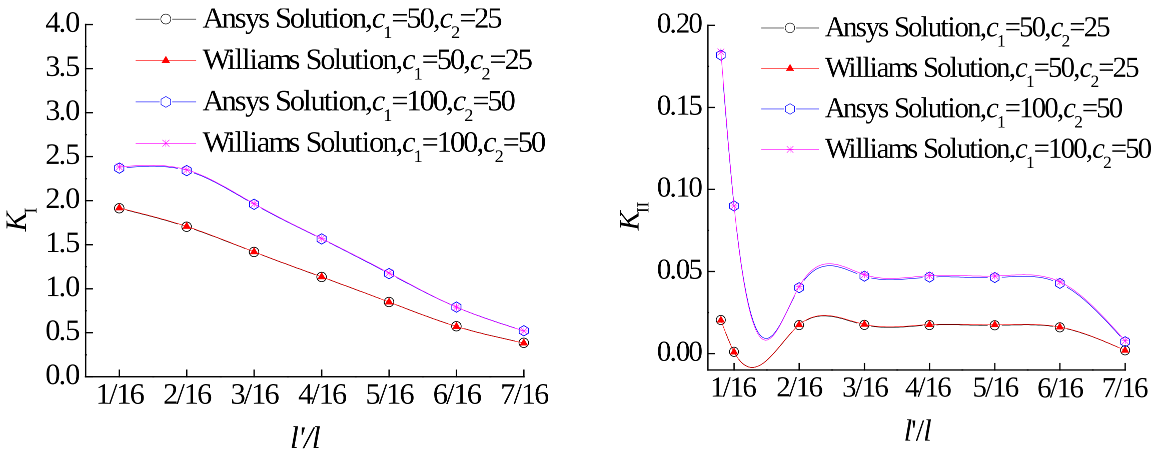

Figure 5. Minor crack K_{I} (left) and K_{II} (right) versus minor cracks' position changed.

Figure 5 shows that with the minor crack furthering from the mid-span, the moment of the section in which the crack position is decreased, thus K_{I} decreases, but near the mid-span, the decreasing trend relatively small. And minor crack K_{II} decreased first and then increased. When the minor crack is very close to the symmetry axis, K_{II} has its maximum value because of the shear mutation in mid-span.

When l'/l smaller than 1/8, minor crack leaves the stress concentration area, K_{II} decreases sharply and then has a certain recover, so it comes to a minimum value. When l'/l is larger than 1/8, the shear remains constant, so the K_{II} also tends to be constant. When the minor crack is near the endpoint, the moment gradually decreases to zero, thus K_{II} decreases to zero.

It can be inferred from the example: When the minor crack relatively close to the main crack of mid-span, the changes of position have a great influence on the main crack tip SIF. And the minor crack tip SIF is relative large, which shows that minor crack and main crack absorb energy together, thus main crack tip K_{I} is smaller, reducing the risk of structural failure, proving the number of crack don't represent the risk of failure. When minor crack relatively far from main crack of mid-span, effect of change in position of the main crack is not obvious, or even negligible. And the minor crack tip of crack K_{I} decreases gradually, K_{II} tends to be constant for a while and decreases to zero when the minor crack is near the endpoint. It shows that with minor crack position changes, K_{I} is affected by the moment and K_{II} is affected by moment and shear.

4.2. Preserve Minor Crack Position with Length Changed (Value C_1=100 Mm; L' =500 Mm, 1000 Mm, 1500 Mm)

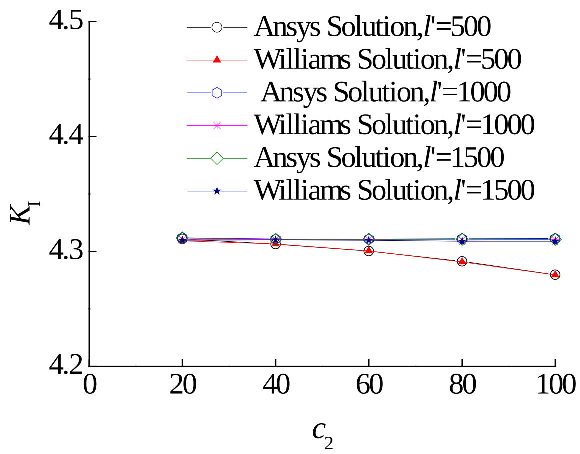

Figure 6. Main crack K_{I} versus minor cracks' length changed.

Figure 6 shows that results of W unit and the singular element calculation are approximately equal, whose relative error is around 1%, and both the changing trends are almost the same. When l '= 1000 mm or 1500 mm, growth with minor crack not be affected the main crack K_{I}. When l '= 500 mm, the main crack K_{I} tended to decreases with minor crack extends.

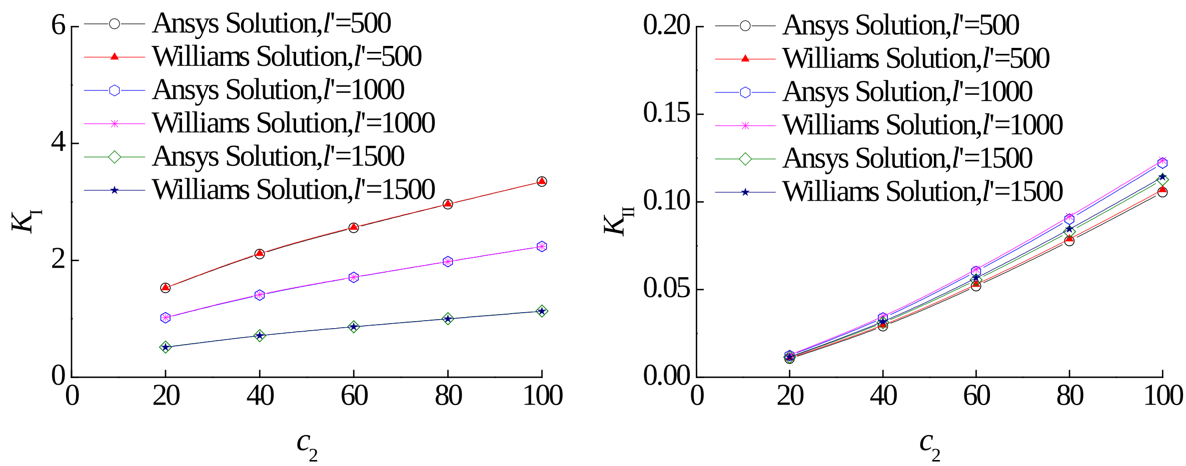

Figure 7. Minor crack K_{I} (left) and K_{II} (right) versus minor cracks' length changed.

Figure 7 shows that the stress concentration becomes more obvious with minor crack extend. Among them, when the minor crack is more near the main crack, the stress concentration of K_{I} is more obvious, but the distance changes are insensitive to K_{II}.

From this example, there comes the following conclusion. The changes of the minor crack length has little influence on the main crack tip SIF, the main crack tip of crack K_{I} will be affected only when length of the minor crack and the main crack are relative equal. The minor crack tip SIF becomes much larger with minor crack extend.

5. Conclusion

This thesis has established the generalized parameter Williams more crack beam element analysis model, analyzed three cracks in three point bending beam crack tip SIF, which shows that results of W unit and the ANSYS finite element solution are identical. This method can obtain SIF directly and has a satisfied accuracy, thus calculation efficiency can be highly improved. W unit shows the strong vitality and can be used widely in the future.

Possible engineering advice is provided according to the results. Crack on the same side of beam in actual project needs major consideration main crack, take main reinforcement measures for main crack. When the minor crack length is short, its impact could be neglected .When length of minor crack and main crack are almost equal, the interaction between main crack and minor crack should be considered.

6. Acknowledgment

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7. References

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