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# Numerical Modelling of Foundation Slabs with use of Schur Complement Method

**Jiří Koktan[1], Jiří Brožovský[1]**

[1]VSB-Technical University of Ostrava, Faculty of Civil Engineering, Ludvíka Podéště 1875/17, CZ- 708 33 Ostrava, Czech Republic.

jiri.brozovsky@vsb.cz

**Abstract**. The paper discusses numerical modelling of foundation slabs with use of advanced numerical approaches, which are suitable for parallel processing. The solution is based on the Finite Element Method with the slab-type elements. The subsoil is modelled with use of Winkler-type contact model (as an alternative a multi-parameter model can be used). The proposed modelling approach uses the Schur Complement method to speed-up the computations of the problem. The method is based on a special division of the analyzed model to several sub-structures. It adds some complexity to the numerical procedures, especially when subsoil models are used inside the finite element method solution. In other hand, this method makes possible a fast solution of large models but it introduces further problems to the process. Thus, the main aim of this paper is to verify that such method can be successfully used for this type of problem. The most suitable finite elements will be discussed, there will be also discussion related to finite element mesh and limitations of its construction for such problem. The core approaches of the implementation of the Schur Complement Method for this type of the problem will be also presented. The proposed approach was implemented in the form of a computer program, which will be also briefly introduced. There will be also presented results of example computations, which prove the speed-up of the solution - there will be shown important speed-up of solution even in the case of on-parallel processing and the ability of bypass size limitations of numerical models with use of the discussed approach.

## 1. Introduction

The correct computational analysis of a building structure has to include an interaction between the structure and the foundation and the subsoil. There are relations between behaviour the structure and the subsoil – deformations of the structure depends on deformations of the subsoil and subsoil deformations and stress state depend on stiffness and deformation of the structure. Thus, the correct solution requires being complex. Foundation structures often have character of slabs and they can be modelled with use of slab finite elements.

The slab computational model is precise enough to capture both stresses and deformations of foundation structure. The selection of an adequate computational model for subsoil is more complex. The obvious approach is a utilization of modern computers and use of 3D models based on the finite elements method [1]. In this case, subsoil is modelled as a relatively large portion of a half space that can be combined with the model of the building structure. This approach was some disadvantages: large 3D models of subsoil result in numerical solutions of very large systems of linear (in some cases even non-linear) equations. It makes this approach relatively computationally intensive. Another problem is the necessity to obtain required input data for the 3D model of subsoil. There are simpler alternative

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approaches that use so-called contact models. These models replace the subsoil by additional constitutive models that make part of the foundation models. This approach greatly reduces size of the computational problem and it allows preparation detailed model of the building structure itself.

The Finite Element Method results in solution of system of linear equations. There is a $n^3$ relation between number of unknown's $n$ and number of required computational operations if any of direct solution methods is used. It obviously results in rapid grow of required computational time for larger problems. In a case of non-linear analysis (and the soil-structure interaction is one of such problems) it is even more important. The computational speed of modern computer processors for single threaded workloads is near constant in the last years, thus the only option to speed-up of solution of this type of problem is parallelization of the solution. The most time-consuming part of the finite element analysis is the solution of systems of linear equations. There can be used one of domain decomposition methods [4]. Those methods use one the main features of the finite element method – an approximation of deformations uses only local approximation functions with short ranges. Use of such functions results in a sparse matrix of the system of linear equations. The numerical model of solved structure thus can be divided to several substructures. Then there will be several smaller linear systems with the total number of unknown, which is equal to the original (non-divided) problem. Such approach results in noticeably shorter computational time even in the case when only single computer processor is used and when these systems are solved in a serial manner.

There are long-term research works on the Faculty of Civil Engineering of the VSB – Technical University of Ostrava that include experimental and numerical works related to soil-structure interaction. This paper discusses one of the pieces of numerical works on this topic. There is studied use of the Schur Complement Method [4] for modelling of foundation slabs on subsoil. The discussed approaches were prepared in the form of the GNU Octave (and the MATLAB-compatible) program code [2].

## 2. Methods of Numerical Analysis

The discussed problem was analysed with use of the Finite Element Method [1] which is arguably the most common numerical method currently used for the problems of continuum mechanics. The main principle of the method is division of the analysed problem domain to finite set of smaller parts – the finite elements. The unknown function is approximated by certain approximation functions. In the problems of continuum mechanics, the unknown function is usually the function of deformations. By using of energy principles (the Lagrange principle, for example [1]) the solution is transformed into a system of linear equations. In this work, there are slabs modelled with use of a nine-node finite element which is based on the Mindlin slab theory [1]. Obviously there are available simpler finite elements for slabs. The selected finite element was chosen after a numerical study [2] was conducted and the chosen element have shown the best behaviour in the studied case. For example, it is less prone to shear locking than four-node element and it makes possible better capture of the internal forces in the studied structure. It of course also eases proper modelling of curved boundaries. A convergence study in [2] has shown that in the soil-structure interaction problem use of the nine-node element also gives much better convergence to an exact solution. The last reason to use the nine-node element was the ability to incorporate the in-plane forces. This possibility is not yet used but it is going to be used in further works as it is expected to use the discussed approaches for modelling of foundations with possibility of horizontal movements.

The effects of subsoil are modelled with use of two-parameter contact model [3]. This model uses two independent constitutive parameters $C_1$ and $C_2$ that are used to describe deformations of subsoil in the contact with the foundation structure. Size of contact stress $\sigma_k$ can be computed from deformation of subsoil $w$:

$$\sigma_k(x,y) = C_1 w(x,y) - C_2 \frac{\partial^2 w(x,y)}{dx^2} - C_2 \frac{\partial^2 w(x,y)}{dy^2}. \tag{1}$$

The subsoil is modelled by dedicated finite elements which use se same approximation functions as the finite elements for slabs. These two types of finite elements can share the same nodes. The difference is in their degrees of freedom: the slab element has three degrees of freedom (unknowns) in every node. There are vertical deformation $w$ and two rotations. The finite element for subsoil has only the one degree of freedom $w$ and it uses different constitutive and geometric equations. These two types of finite elements share the same nodes in the area of foundation slab. To properly capture shear forces in the subsoil it is necessary to extend the modelled area of subsoil beyond the foundation geometry. The size of such extension depends on the ratio of the $C_2$ and $C_1$ parameters. The shared nodes between subsoil and foundation models not only guarantee consistent deformations but they also allow transfer of the effects of vertical forces from the structure to the subsoil. The effects of horizontal forces are not used in the discussed model. It is assumed here that foundation and subsoil have common deformations only of there are compression between foundation and subsoil. It there is a tension then the foundation and subsoil behave independently. Thus, the solution is non-linear and iterative procedure is used.

## 3. Schur's Complement Method

The Schur complement method is also known as the static condensation method [4]. It is based on division of the finite element model to several sub-structures which share boundary nodes. Every sub-structure has its set of internal nodes. Size and number of these sub-structures can be arbitrary. For effective solution, it is recommended to have sub-structures, which incorporate identical numbers of internal nodes and to have as small number of boundary nodes as possible. An example of an effective division of a rectangular slab is shown in the Figure 1.

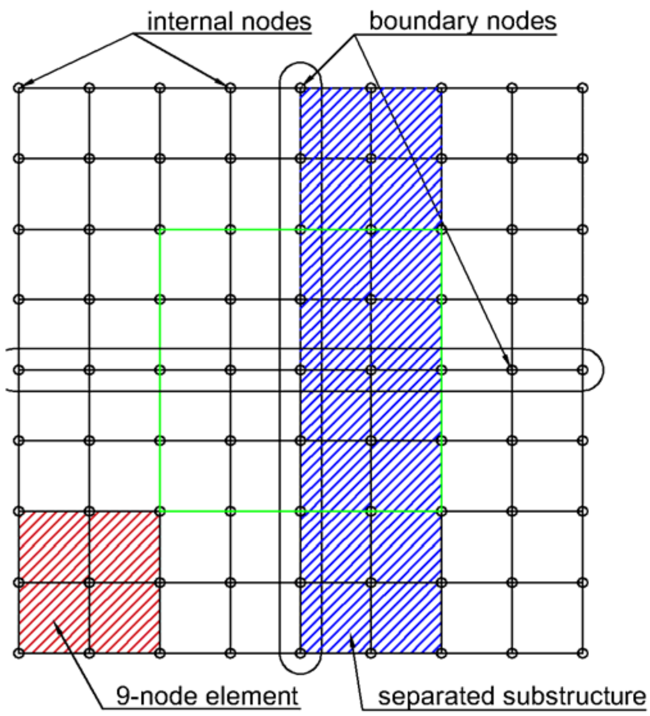


Figure 1: Illustration of domain decomposition into four substructures. Green line represents the edge of foundation slab

The practical use of the Schur Complement Method requires certain system in numbering of internal and boundary nodes [4]. The numbering of nodes has influence on the structure of stiffness matrices of substructures and on the structure of matrix of the whole problem. The numbering procedure requires numbering the internal nodes of individual sub-structures first. Then the boundary nodes are numbered. Then is is possible to construct the linear system of the problem of m+1 sub-matrices in the form of (2).

$$\begin{bmatrix} K_1^{ii} & 0 & 0 & \cdots & 0 & K_1^{ib} \\ 0 & \ddots & 0 & & & 0 \\ 0 & 0 & K_j^{ii} & & & K_j^{ib} \\ \vdots & & & \ddots & 0 & 0 \\ 0 & & & 0 & K_m^{ii} & K_m^{ib} \\ K_1^{bi} & 0 & K_j^{bi} & 0 & K_m^{bi} & K^{bb} \end{bmatrix} \cdot \begin{bmatrix} r_1^i \\ \vdots \\ r_1^i \\ \vdots \\ r_m^i \\ r^b \end{bmatrix} = \begin{bmatrix} F_1^i \\ \vdots \\ F_1^i \\ \vdots \\ F_m^i \\ F^b \end{bmatrix}, \tag{2}$$

where the $K$ is a stiffness matrix, $r$ is a vector of unknown deformations and $F$ is a vector of forces in nodes. The $j$ index contains a number of a sub-domain, the $i$ index contains a number of internal part of a sub-structure and the $b$ defines the boundary parts of vectors and matrices. All $K_j^{ii}$ matrices are regular and it is possible to invert them. The expression of deformations of internal nodes $r_j^i$ and their use in the last line of matrix equation (2) can result in a reduced system of equations (3).

$$\left( \left[K^{bb}\right] - \sum_{j=1}^{m} \left[K_j^{bi}\right] \cdot \left[K_j^{ii}\right]^{-1} \cdot \left[K_j^{ib}\right] \right) \cdot \left\{r^b\right\} = \left\{F^b\right\} - \sum_{j=1}^{m} \left[K_j^{bi}\right] \cdot \left[K_j^{ii}\right]^{-1} \left[F_j^i\right]. \tag{3}$$

The bracketed part of the Equation (3) is the Schur Complement of the sub-matrix $K^{bb}$. The size of the reduced problem (3) depends of the number of unknowns in the boundary nodes. When the problem (3) is solved then it is possible to computer unknowns in internal nodes with use of the Equation (2). In this phase, this equation can be divided to $m$ independent problems for $m$ sub-structures. Size of a linear equations system of every sub/problem depends on number of internal unknowns in an individual sub-structure.

The equations above are in the form which is appropriate for description of the method but bot for its practical application and it requires some modifications. The main advantage of the method is the fact that matrices of sub-structures are independent and that in any part of solution it is not necessary to construct all of them in the same time. This fact was used when the computer program was prepared and our implementation of the method requires no special numbering of structural nodes. The only input requirement is the division of the problem to the sub-structures. These data are used by the program for a construction of two systems of node numbering. The first system is created in the same manner as in a usual non-parallel finite element method code. It means than code numbers of individual degrees of freedom are based on a nude number and on a number of degrees of freedom in each node. Then a local and substructure-specific numbering of degrees of freedom is prepared. This system starts from number one in any sub-structure. The numbering of the degrees of freedom of boundary nodes is also independent. There are also computed date that are used to mapping of the local numbering to the global numbering of degrees of freedom. The construction of stiffness matrices members of sub-structures is done in a cycle over all sub-areas to fulfil the Equations (2). The data computed for every sub-structure are also used to create the sums in the Equation (3). There is no necessary order of these sub-structures so this part can be executed in a serial manner or in parallel. When all matrices are constructed then it is possible to compute the degrees of freedom in the boundary nodes with use of the Equation (3). Then it is possible to compute internal unknowns in every sub-domain. Also this part can be executed in parallel. The last step is conversions of the results to global numbering and analysis of the results.

## 4. Numerical Example

A rectangular reinforced concrete slab of dimensions 2x2m on an elastic foundation has been studied. The thickness of the slab wa 0.15 m The slab was loaded by a single force $F_Z$=300kN in the center. For the purpose of this example a linear elastic and isotropic material model was considered. There was used Young modulus $E$ = 34 GPa a and Poisson ratio $\upsilon$=0.2. The parameters of subsoil have been obtained from experiment and they were: $C_1$ = 300MNm$^{-3}$, $C_2$ = 10MNm$^{-1}$ . This numerical example is

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a simplified case of the experiment conducted at the Faculty of Civil Engineering of the VSB – Technical University of Ostrava in 2016. The experiment is illustrated on the Figure 2.

Figure 2: Experimental slab test

The numerical analysis was conducted for several cases with different densities of finite element meshes. A basic form of the finite element method and the Schur Complement Method in the form described above was used. The problem was divided to the four sub-structures as it is illustrated in Figure 1. The advantage of this division is that the number of internal nodes grows faster than number of boundary nodes and thus the size of the reduced problem remains relatively small even for relatively dense meshes.

## 5. Results and Discussion

A comparison of effectivity of the basic form of the finite element solution (called Full SLE in the Figure 3) and the Schur Complement Method (Decomposition) was done. In the Figure 3 there are shown computational times for several finite elements mesh densities. The times in the Figure 3 are given for a single iteration of the non-linear solution, as the use of the Schur Complement Method had no negative effect on the iterative solution. If the problem was small enough then the Schur Complement method required, the same or even slightly greater computational time. This slowdown was caused by additional operations necessary for handling of sub-structures. However, for larger problems it is obvious that the Schur Complement Method became much faster than the basic form of the finite element solution. The incomplete data were caused by the limitations of used hardware (there was reached a random access memory size limit) – the basic form of the solution resulted in too large linear system which became outside limits but the Schur Complement Method produced linear systems small enough to me still solvable. However, even for the comparable data size the Schur Complement Method was noticeably faster. It is good to mention that no parallel hardware was used for this test – the speedup was reached solely by using of a more advanced computational procedure.

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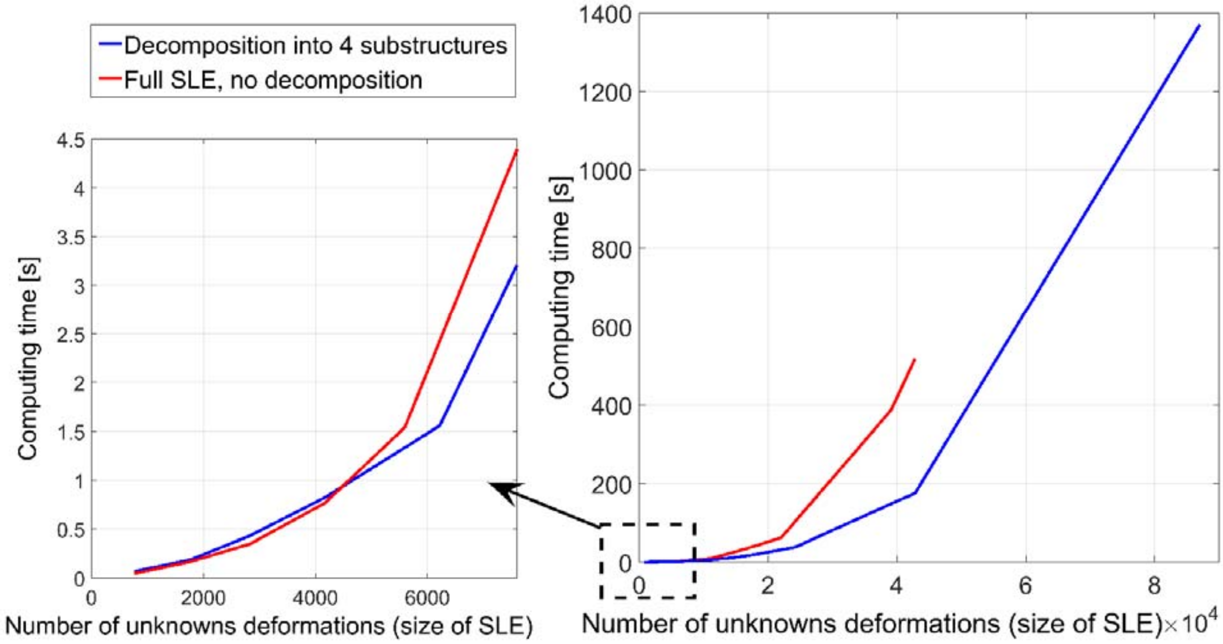


Figure 3: Computing time required to perform one iteration step of FEM analysis in relation to the size of the system of linear equations SLE

In a case of no-linear problems the number of iterations can multiply the saved time and thus the speed-up became even more noticeable. The studied slab, which is loaded in its center, has the tendency raise its edges as it is shown in Figure 4. Thus an iterative solution had to be used here. In the first iteration, the deformations of the slab and the subsoil were consistent but at the end of the solution, there were computed deformations shown in the Figure 4. These shown results were obtained for the problem with 35 293 unknowns. There were 11 iterations and solution time was 10 minutes. For the basic form of the finite element solution, the time of single iteration was 5 minutes and the whole solution required more than one hour.

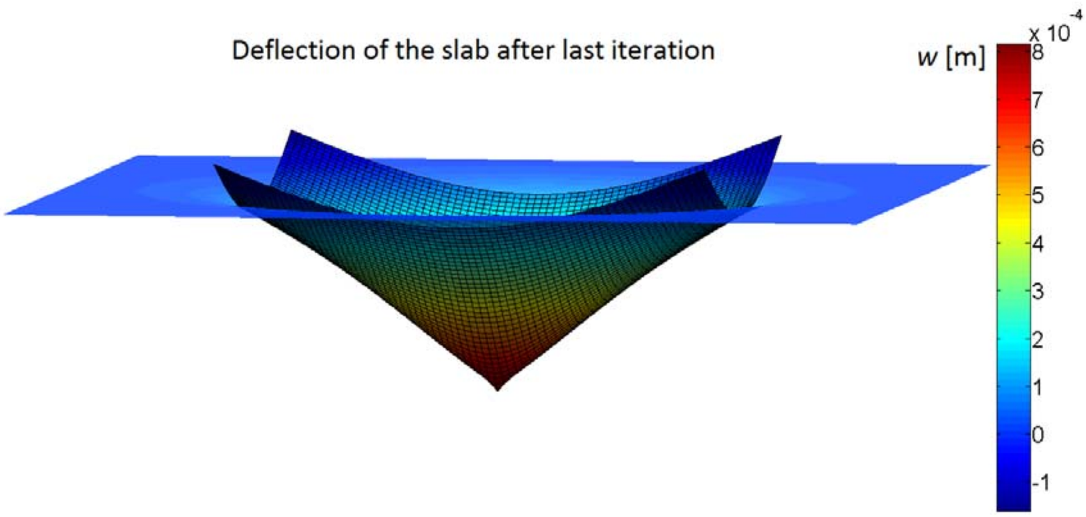


Figure 4: Deflection of the slab

## 6. Conclusions

The paper discussed use of the Schur Complement Method for computational analysis of foundation slabs on subsoil. It was shown that the method can be used here and it can shorten the computational time and it allows to solve larger problems than usual finite element method solution procedures. The presented program code did not utilize parallel features of computer hardware but its improvements will be a part of further works. The parallelization of solution is possible as there are independent parts of solution that can be executed in parallel.

The used model of subsoil has certain limitations. To be able to capture the shear effects in the subsoil it is necessary to extend the modelled area of subsoil beyond the boundaries of the foundation structure. This extension introduces further unknown into the solution. Another problem is that the model combines two different finite element types with different numbers of degrees of freedom in their nodes. It has some impact to the cleanness of the program and then to the computational speed. The subsoil model with two parameters is known to be adequate in some cases but it is not general enough. These problems can be resolved for example by use of the model described in [5], which uses only one parameter $C_1$. This parameter is not used as a constant but it has a form of a continuous function for whole area of the foundation. The parameters of the function are computed by a numerical procedure in the integration points of finite elements on the basis of a modified equations of an elastic half-space. Such approach makes possible to eliminate the need of two types of finite elements in the model. This approach is going to be included in the discussed program code and it is going to be used for further works.

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