

Applications of polynomial optimization in financial risk investment

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Abstract. Recently, polynomial optimization has many important applications in optimization, financial economics and eigenvalues of tensor, etc. This paper studies the applications of polynomial optimization in financial risk investment. We consider the standard mean-variance risk measurement model and the mean-variance risk measurement model with transaction costs. We use Lasserre's hierarchy of semidefinite programming (SDP) relaxations to solve the specific cases. The results show that polynomial optimization is effective for some financial optimization problems.

1. Introduction

Financial risk refers to the uncertainty of investment results that is affected by one or more random factors, including operational risk, credit risk, exchange rate risk and market risk. The greater the risk is, the greater the return or loss is. To remain invincible in the face of risk, it is necessary to implement the risk management. The financial risk management has its strategic significance, especially in market deregulation, globalization and cross business today. Quantification and control of risk has become the core of all the activities of investors and financial regulatory authorities. To manage risk, we need to quantify or measure the size of risk. After more than half a century of development, the financial risk measurement methods have made the great progress. Especially with the successful use of complex mathematical methods, there have been a lot of good measurement models in recent years, so the research on the risk measurement has become a hot research field of Applied Mathematics.

In 1952, Markowitz first gave a quantitative method of selecting the optimal portfolio, which is known as the mean-variance model. In this model, he proposed to use the mathematical expectation $E[X]$ of portfolio random income X to represent its expected return, and use the variance $Var(X) = E[(X - E[X])^2]$ to measure the risk of portfolio. The Markowitz method is easy to achieve, because $Var(X)$ can be calculated through the covariance, which is based on the joint distribution of the return of each asset. This model and the inherent risk decentralization thought become the foundation of modern investment theory.

Markowitz portfolio theory has become an important tool to help investors how to invest capitals in a certain proportion, which makes the investment income big and small risk. $Var(X)$ is generally a nonconvex function, and there are many local extremums, so the optimal portfolio choice problems are very difficult to use the usual optimization algorithm [10] for control and optimization, and results may not be stable. It is precisely this reason that only a few literatures [1, 3, 6] really use Var as a risk measure for the actual large-scale investment portfolio optimization problems.



In recent years, polynomial optimization theory has made great progress (see [5, 7-9]), which can find the global optimal point from some local extremums for the nonconvex polynomial functions. In this paper, we construct the standard mean-variance risk measurement model and the mean-variance risk measurement model with transaction costs. We use Lasserre's hierarchy of SDP relaxations to solve two cases in financial risk investment, and the results show that polynomial optimization can effectively solve some actual financial optimization problems.

2. Polynomial optimization theory

In this section, we introduce the preliminary knowledge of polynomial optimization (see [5, 7, 8]).

Let \mathbb{N}, \mathbb{R} be the sets of nonnegative integer numbers and real numbers, respectively. The symbol $\mathbb{R}[x] := \mathbb{R}[x_1, x_2, \dots, x_n]$ denotes the polynomial ring in $x := [x_1, x_2, \dots, x_n]$ with real coefficients. For a degree d , $\mathbb{R}[x]_d$ denotes the set of all polynomials in $\mathbb{R}[x]$ whose degrees are at most d . Let $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \alpha_2 + \dots + \alpha_n \leq d\}$. The symbol $\deg(p)$ denotes the degree of polynomial p . For a positive integer k , $[k] = \{1, 2, \dots, k\}$.

An ideal I in $\mathbb{R}[x]$ is a subset of $\mathbb{R}[x]$ such that $I \cdot \mathbb{R}[x] \subseteq I, I + I \subseteq I$. For a tuple $h = (h_1, \dots, h_m)$ in $\mathbb{R}[x]$, $\langle h \rangle$ denotes the smallest ideal containing all h_i , i.e. $\langle h \rangle = h_1 \cdot \mathbb{R}[x] + \dots + h_m \cdot \mathbb{R}[x]$. The k -th truncation of the ideal $\langle h \rangle$ is denoted by $\langle h \rangle_k$, which is the set

$$\langle h \rangle_k = h_1 \cdot \mathbb{R}[x]_{k-\deg(h_1)} + \dots + h_m \cdot \mathbb{R}[x]_{k-\deg(h_m)}.$$

A polynomial σ is called a sum of squares (SOS) if $\sigma = p_1^2 + \dots + p_k^2$ for some $p_1, \dots, p_k \in \mathbb{R}[x]$. $\Sigma[x]$ denotes the set of all SOS polynomials in x . For a degree m , $\Sigma[x]_m$ denotes the truncation $\Sigma[x] \cap \mathbb{R}[x]_m$. For a tuple $g = (g_1, \dots, g_t)$, its quadratic module is the set

$$Q(g) = \Sigma[x] + g_1 \cdot \Sigma[x] + \dots + g_t \cdot \Sigma[x].$$

The k -th truncation of $Q(g)$ is the set

$$Q_k(g) = \Sigma[x]_{2k} + g_1 \cdot \Sigma[x]_{d_1} + \dots + g_t \cdot \Sigma[x]_{d_t},$$

Where $d_i = 2k - \deg(g_i), i = 1, 2, \dots, t$. For polynomial tuples h and g , the sum $\langle h \rangle + Q(g)$ is called archimedean if there exists $p \in \langle h \rangle + Q(g)$ such that $p(x) \geq 0$ defines a compact set $\mathbb{R}[x]$. Suppose that $p \in \langle h \rangle + Q(g)$ is archimedean, if a polynomial f is positive on the set $\{h(x) = 0, g(x) \geq 0\}$, then $f \in \langle h \rangle + Q(g)$ (see Putinar [11]).

Let \mathbb{R}_d^n represent the space of real sequences indexed by $\alpha \in \mathbb{N}_d^n$. A vector y in \mathbb{R}_d^n is called a truncated moment sequences (tms) of degree d , i.e.

$$y = (y_\alpha)_{\alpha \in \mathbb{N}_d^n}.$$

A tms $y \in \mathbb{R}_d^n$ defines a Riesz function L on $\mathbb{R}[x]_d$ as

$$L_y\left(\sum_{\alpha \in \mathbb{N}_d^n} p_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}\right) := \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha y_\alpha.$$

For convenience, let $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\langle p, y \rangle := L_y(p)$.

Let $q \in \mathbb{R}[x]$. For every $y \in \mathbb{R}_{2k}^n$, the function $L(qp^2)$ defines as

$$L(qp^2) = \text{vec}(p)^T (L_q^k(y)) \text{vec}(p),$$

Where $\text{vec}(p)$ denotes the coefficient vector of polynomial p with $\deg(qp^2) \leq 2k$, the symmetric matrix $L_q^k(y)$ is the k -th localizing matrix of q , generated by y , which is linear about y . For example, $n = 2, k = 2, q = 1 - x_1^2 - x_2^2$, it follows

$$L_{1-x_1^2-x_2^2}^{(2)}(y) = \begin{pmatrix} 1 - y_{20} - y_{02} & y_{10} - y_{30} - y_{12} & y_{01} - y_{21} - y_{03} \\ y_{10} - y_{30} - y_{12} & y_{20} - y_{40} - y_{22} & y_{11} - y_{31} - y_{13} \\ y_{01} - y_{21} - y_{03} & y_{11} - y_{31} - y_{13} & y_{02} - y_{22} - y_{04} \end{pmatrix}.$$

When $q = 1, L_1^{(k)}(y)$ is called the k -th moment matrix generated by y , denoted by $M_k(y)$. For instance, $n = 2, k = 2$,

$$M_2(y) = \begin{pmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{pmatrix}.$$

For the polynomial optimization problem

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } & h_1(x) = \dots = h_{m_1}(x) = 0, \\ & g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0, \end{aligned} \quad (2.1)$$

Where $f(x), h_i(x), g_j(x)$ are polynomial functions. Let u be a local minimizer of (2.1), $J(u) = \{j_1, \dots, j_r\}$ be the active set at u . We can construct new polynomials $\varphi_1, \dots, \varphi_r$, such that (2.1) is equivalent to the following problem

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} f(x) \\ \text{s.t. } & h_i(x) = \varphi_j(x) = 0, i \in [m_1], j \in [r] \\ & g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0. \end{aligned} \quad (2.2)$$

For solving (2.2), the hierarchy of semidefinite programming (SDP) relaxations (see[5]) is

$$\begin{cases} \rho_k = \min \langle f, y \rangle \\ \text{s.t. } L_{h_i}^{(k)}(y) = 0, L_{\varphi_j}^{(k)}(y) = 0, i \in [m_1], j \in [r] \\ M_k(y) \succeq 0, L_{g_i}^{(k)}(y) \succeq 0, i = 1, 2, \dots, m_2 \end{cases} \quad (2.3)$$

The dual problem of (2.2) is

$$\eta_k := \max \gamma \quad \text{s.t.} \quad f - \gamma \in \langle h \rangle_{2k} + Q_k(g). \quad (2.4)$$

Denote by

$$f_{\min} := \min f(x), \text{s.t. } h(x) = 0, g(x) \geq 0,$$

Where $h = (h_1, \dots, h_{m_1})$, $g = (g_1, \dots, g_{m_2})$. For $k = 1, 2, \dots$, solving the SDP (2.4), we get a sequence with lower bounds

$$\eta_1 \leq \eta_2 \leq \eta_3 \leq \dots \leq f_{\min}.$$

Theorem 1 ([5]) If $\langle h \rangle + Q(g)$ is archimedean, then

$$\lim_{k \rightarrow \infty} \eta_k = f_{\min}.$$

In fact, when k is big enough, Lasserre's hierarchy generally has finite convergence (see Theorem 1.1 in [8]). The software Gloptiploy 3 [4] can solve (2.3) and (2.4).

3. Optimization models in financial risk investment

According to the demands of different style investors, we can construct different optimization models with the Markowitz theory.

3.1 Standard mean-variance risk measurement model

With n kinds of securities available for investment options, note that the yield of the i -th security during the j -th period is r_{ij} , $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, the proportion of the total assets of the i -th security is x_i , $i = 1, 2, \dots, n$, the average yield and its variance of the m -th period of the i -th security are

$$\bar{r}_i = \frac{1}{m} \sum_{j=1}^m r_{ij}, \sigma_i^2 = \frac{1}{m} \sum_{j=1}^m (r_{ij} - \bar{r}_i)^2,$$

the covariance of the i -th and j -th securities' yield is

$$\sigma_{ik} = \frac{1}{m} \sum_{j=1}^m (r_{ij} - \bar{r}_i)(r_{kj} - \bar{r}_k).$$

The investment risk can be defined as

$$V = \sum_{i=1}^n \sigma_i^2 x_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sigma_{ij} x_i x_j. \quad (3.1)$$

Denote $x = (x_1, x_2, \dots, x_n)^T$ as the portfolio vector, then the above investment risk V can be expressed as $V = x^T Q x$, where

$$Q = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}, \sigma_{ij} = \sigma_i^2, \sigma_{ik} = \sigma_{ki}, i = 1, 2, \dots, n, j = 1, 2, \dots, n,$$

is a symmetric and semidefinite matrix.

Denote the yield vector as $\bar{r} = (\bar{r}_1, \bar{r}_2, \dots, \bar{r}_n)^T$, then the investment yield can be expressed as

$$R = \sum_{i=1}^n \bar{r}_i x_i = x^T \bar{r}.$$

In the case of allowing the short-term trading, if the decision is to minimize risk, then the risk model is obtained as follows

$$\min V = x^T Q x, \quad \text{s.t.} \quad \sum_{i=1}^n x_i = 1.$$

The above model is only aware of the risk, not considering the return. In the optimization problems of financial portfolio, if different conditions are considered, the optimization model can be different.

For instance, considering how to purchase a number of assets from n kinds of risk assets, the variance risk is as small as possible under the premise of the yield to a certain level. Based on the above conditions, we can construct the following model [2]

$$\left\{ \begin{array}{l} \min \quad V = x^T Q x, \\ \text{s.t.} \quad \sum_{i=1}^n x_i = 1, \\ \quad \quad \sum_{i=1}^n \bar{r}_i x_i = r_p, \\ \quad \quad x_i \geq 0, i = 1, 2, \dots, n, \end{array} \right. \quad (3.2)$$

Where r_p is the expected return of investment portfolio.

3.2 Mean-variance risk measurement model with transaction costs

Transaction cost generally refers to the amount of money that is paid for the purchase of an asset more than its present value, which is an important factor that investors must consider when choosing a portfolio.

With n kinds of risk assets available for selection, note that the mean value of the return of the i -th asset is $\bar{r}_i, i = 1, 2, \dots, n$, the covariance matrix is $Q = (\sigma_{ij})$, where σ_{ij} is the covariance of the returns of the i -th and j -th assets. The amount of the purchase of the i -th asset is x_i , the transaction cost is denoted by $c_i(x_i)$. How to purchase a number of assets from the n kinds of assets, the variance risk is as small as possible under the premise of the yield to a certain level. Based on the above conditions, we also can construct the following model [12]

$$\left\{ \begin{array}{l} \min \quad V = x^T Q x, \\ \text{s.t.} \quad \sum_{i=1}^n x_i = 1, \\ \quad \quad \sum_{i=1}^n (\bar{r}_i x_i - c_i(x_i)) = r_p, \\ \quad \quad x_i \geq 0, i = 1, 2, \dots, n, \end{array} \right. \quad (3.3)$$

Where r_p is the expected return of investment portfolio. The cost function $c_i(x)$ is not given, and is generally a nonlinear function. $c_i(x)$ is generally obtained by using least squares fitting based on the actual data.

4. Applications

In this section, we use polynomial optimization to solve the financial risk investment cases.

Case 1. An investor intends to choose three stocks A, B, C with outstanding performance to carry out the long-term portfolio investment. Through the market analysis and statistical forecast of three stocks, we get the relevant data shown in Table 1. Please give the investment proportion of three stocks from two aspects.

(1) Such that the variance of stock returns in the portfolio is reduced to a minimum, to reduce the risk of investment, and hope that the expected yield after five years is less than 65%.

(2) Hope to get the maximum benefit, in the case that the standard deviation is not more than 12%.

Table 1. Related data of stocks.

Name of stocks	Expected yields in five years%	Covariance in five years %		
		A	B	C
A	92	180	36	110
B	64	36	120	-30
C	41	110	-30	140

Let x_1, x_2, x_3 respectively represent the investment ratio of three shares A, B, C. Denote the expected yield in five years as r_1, r_2, r_3 , which are random variables. The total return of the portfolio in five years is

$$R = x_1 r_1 + x_2 r_2 + x_3 r_3,$$

the variance of the portfolio is

$$\text{var}(R) = x_1^2 \text{var}(r_1) + x_2^2 \text{var}(r_2) + x_3^2 \text{var}(r_3) + 2x_1 x_2 \text{cov}(r_1, r_2) + 2x_1 x_3 \text{cov}(r_1, r_3) + 2x_2 x_3 \text{cov}(r_2, r_3).$$

Based on the data of Table 1, the portfolio risk is obtained

$$V = \text{var}(R) = 180x_1^2 + 120x_2^2 + 140x_3^2 + 72x_1x_2 + 220x_1x_3 - 60x_2x_3.$$

Therefore, the standard deviation of the portfolio is

$$D = [180x_1^2 + 120x_2^2 + 140x_3^2 + 72x_1x_2 + 220x_1x_3 - 60x_2x_3]^{\frac{1}{2}}.$$

According to the conditions of item (1), we construct the mathematical model of the problem

$$\begin{cases} \min V = 180x_1^2 + 120x_2^2 + 140x_3^2 + 72x_1x_2 + 220x_1x_3 - 60x_2x_3, \\ \text{s.t. } x_1 + x_2 + x_3 = 1, \\ \quad 0.92x_1 + 0.64x_2 + 0.41x_3 \geq 0.65, \\ \quad x_1, x_2, x_3 \geq 0. \end{cases} \quad (4.1)$$

Using Lasserre's hierarchy of SDP (2.3) and (2.4) to solve, we get

$$x_1 = 0.2351, x_2 = 0.5222, x_3 = 0.2427,$$

the optimal solution of the objective function is $V = 64.7054$. In other words, under the conditions of ensuring the minimum risk and the total yield in five years being 65%, the portfolios of three stocks A, B, C are 23.51%, 52.22%, 24.27%, the minimum variance is 0.6471%.

According to the conditions of item (2), we construct the mathematical model of the problem

$$\begin{cases} \max R = 0.92x_1 + 0.64x_2 + 0.41x_3, \\ \text{s.t. } x_1 + x_2 + x_3 = 1, \\ \quad 180x_1^2 + 120x_2^2 + 140x_3^2 + 72x_1x_2 + 220x_1x_3 - 60x_2x_3 \leq 144, \\ \quad x_1, x_2, x_3 \geq 0. \end{cases} \quad (4.2)$$

Using Lasserre's hierarchy of SDP (2.3) and (2.4) to solve, it follows

$$x_1 = 0.8593, x_2 = 0.1407, x_3 = 0.0000,$$

the optimal solution of the objective function is $R = 0.8806$. In other words, under the conditions that the standard deviation is not more than 12% and the maximum income is obtained in five years, the portfolios of three stocks A, B, C are 85.93%, 14.07%, 0%, the highest yield can reach 88.06%.

Case 2. An investor intends to choose five securities A, B, C, D, E to carry out the portfolio investment. Through the market analysis and statistical forecast of five securities, we obtain the relevant data shown in Table 2. The functions of transaction costs are defined as

$c_i(x_i) = 0.004x_i - 0.001x_i^2, i = 1, 2, \dots, 5$. Please give the investment proportion of five securities, such that the expected yield is 10% and the risk of investment is the minimum.

Table 2. Related data of securities.

Name of securities	The average yields	Covariance of yields				
		A	B	C	D	E
A	0.16	0.56	0.11	0.09	0.08	0.13
B	0.11	0.11	0.32	0.20	0.14	0.11
C	0.08	0.09	0.20	0.22	0.13	0.12
D	0.13	0.08	0.14	0.13	0.48	0.11
E	-0.02	0.13	0.11	0.12	0.11	0.24

Let x_1, x_2, x_3, x_4, x_5 respectively represent the investment ratio of five securities A, B, C, D, E. This problem is an investment portfolio with transaction costs. According to the optimization model (3.3), we construct the mathematical model of the problem

$$\left\{ \begin{array}{l} \min V = 0.56x_1^2 + 0.32x_2^2 + 0.22x_3^2 + 0.48x_4^2 + 0.24x_5^2 + 0.22x_1x_2 + 0.18x_1x_3 + 0.16x_1x_4 \\ \quad + 0.26x_1x_5 + 0.4x_2x_3 + 0.28x_2x_4 + 0.22x_2x_5 + 0.26x_3x_4 + 0.24x_3x_5 + 0.22x_4x_5, \\ \text{s.t. } \sum_{i=1}^5 x_i = 1, \\ \\ 0.16x_1 + 0.11x_2 + 0.08x_3 + 0.13x_4 - 0.02x_5 - 0.004\sum_{i=1}^5 x_i + 0.001\sum_{i=1}^5 x_i^2 = 0.1, \\ \\ x_i \geq 0, i = 1, 2, \dots, 5, \end{array} \right.$$

Using Lasserre's hierarchy of SDP relaxations (2.3) and (2.4) to solve, we obtain

$$x_1 = 0.2191, x_2 = 0.1156, x_3 = 0.4110, x_4 = 0.1878, x_5 = 0.0666.$$

The optimal solution of the objective function is $V=0.1746$. Under the condition that the expected yield is 10%, the portfolios of five securities are 21.91%, 11.56%, 41.10%, 18.78%, 6.66%, the minimum risk is 17.46%.

The above cases show that polynomial optimization can effectively solve some actual financial optimization problems.

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