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Mathematical modeling of capillary flow of viscous fluid

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Abstract. We consider flow of viscous fluid in the capillaries under assumption that the contact angle of the curve varies over the length of the capillary. For the two laws of change we obtain estimates of the distance of meniscus from its initial position and reduction of its movement speed. It was found that the results of the evaluations are consistent with the existing physical concepts.

1. Introduction

Fluid flow in capillaries (capillary flow) is often found in nature and technology; it plays an important role in the plant and animal life, when filtering the fluid media in the grounds, in the drying processes of dispersed bodies, media purification from mechanical impurities, impregnation of porous materials and in many other cases. Fundamental principles of capillary fluid flow theory are presented in [1-6 and etc.]. Let’s name some of them.

In general case, the fluid transfer in the capillaries occurs by both molecular (diffusive) and convective mechanisms. First is caused by the difference in concentration of a substance along the length of the capillary. It is important to note that because of braking action of the capillary solid walls the diffusion coefficient is much smaller than that in a free fluid and this fluid transfer mechanism will be the most significant in very thin capillaries.

When capillary lateral dimensions are relatively large the convective transfer is dominating its intensity is determined, firstly, by the pressure drop along the length of the capillary, secondly, by the temperature gradient along it, thirdly, by the action of the surface tension forces at the interface between fluid and gaseous medium (meniscus), mass forces, in particular, gravitational forces.

The surface tension force is the force acting on a contour unit length of the media interface and is characterized by the surface tension coefficient σ also called surface tension. The magnitude of the surface tension coefficient depends on the material of media in contact, temperature, presence of adsorbed (surface active) agent on the fluid surface and other factors.

2. Part of mathematical modeling

Taking into account the surface tension, in the absence of mass forces at the contact boundary of immiscible viscous fluid and gaseous medium the following relations are written [3]:

$$p_{nnl} + p_\sigma = p_{nng}, p_{nsl} + p_s = p_{nsg}, \tag{1}$$

They characterize the balance of loads acting on the interface surface;

$$v_{nl} = v_{ng} = w_n, v_{sl} = v_{sg} = w_s.$$

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Here, the subscripts l, g mean the parameters of the fluid and gaseous medium, respectively; p_{nn}, p_{ns} – normal and tangential components of stress $\vec{p}_n$; v_n, v_s; w_n, w_s – components of motion velocity of medium, interface surface; $p_\sigma, p_s = (grad\ \sigma)_s$ – capillary pressure, tangential force per unit area.

In the simplest case, when $p_{nnl} \approx -p_l$,$p_{nng} \approx -p_g$, the surface tension along the contact boundary is constant $((grad\sigma)_s = 0)$, the capillary – tube of radius r, meniscus has a shape of a spherical segment, capillary pressure [5]

$$p_\sigma = \pm 2\sigma/\tilde{R}.$$

Here, $\tilde{R}$ – mean radius of the interface curvature, $\tilde{R} = r/\cos\theta$, θ – limiting wetting angle; plus sign is selected for a convex fluid surface, minus sign – for a concave $(p_\sigma < 0)$.

In accordance with the Young's equation [4-6]

$$\cos\theta = \left(\sigma_{sg} - \sigma_{sl}\right)/\sigma, \tag{2}$$

where σ_{sg}, σ_{sl} – surface tension at the interfaces solid body – gas, solid body – fluid.

In work [6] it is shown that the contact angle determined according to (2) gives the angle value at the three phase boundary of the media in contact on a microscopic scale. The macroscopic contact angle depends on the system geometry which is, in turn, under a great influence of the mass forces, structure and material of the solid body surface, medium temperature.

When studying the capillary flow of fluid in the above-mentioned cases it is usually considered that capillary is a tube, probably, with varying along the length cross-section [5], the media in a capillary are under action of gravitational force, along the movement of meniscus the limiting wetting angle remains constant.

Analyzing the available result you can conclude that during the spraying of polymeric powder materials [7-10] at the stage of coalescence of fluid particles situated on the treated body surface, their spreading [8], the medium movement is similar to the capillary one. For better understanding of peculiarities of the given processes taking into account the specificity of particles coalescence we consider the capillary flow of fluid between two plates situated at a small distance from each other $d = 2r$. Let's assume that the meniscus cross-section is close to a circular arc of radius R its surface is under the influence of pressure p_g from the gas side, pressure p_P – fluid. Let's also assume that during the meniscus movement the limiting wetting angle varies according to a particular law.

Assuming that the surface of the channel walls is wettable the contact angle $0 < \theta < \pi/2$, we represent the dependence $\zeta = \cos\theta$ in the form:

$$\zeta = \zeta(\bar{h}) = 1 - (1 - \bar{\zeta}_K) \cdot \bar{h}. \tag{3}$$

Here $\bar{h} = h/h_K$,$h = h(\tau)$ – meniscus distance from the original position occupied by it in the capillary at the time $\tau = 0$;

$$h_K = \sigma\zeta_K/\Delta p_c^*,\ \zeta_K = \cos\theta_K,$$

θ_K – characteristic limiting wetting angle for the considered system solid body-fluid-gas, parameter $\bar{\zeta}_K \geq \zeta_K$, $\Delta p_c^* = p_g - p_p^*$ – pressure drop of media in contact on the meniscus surface.

The average velocity of the meniscus movement will be:

$$\tilde{v}_M = dh/d\tau = h_K \cdot d\bar{h}/d\tau.$$

Considering as it is customary in the capillary hydrodynamics that the given movement of fluid is laminar, close to the stationary (quasi-stationary) to describe the average velocity of its movement in the absence of the mass forces we use the Poiseuille solution:

$$\tilde{v}_\Pi = r^2(p_\sigma - \Delta p_c)/\left(3\eta h_K \bar{h}\right),$$

where η – fluid viscosity, $\Delta p_c = p_g - p_p$, when it flows plane-parallel between the plates

$$p_\sigma = \sigma\zeta\left(\bar{h}\right)/r.$$

Equating the velocities $\tilde{v}_M$ and $\tilde{v}_\Pi$, after a series of transformations we obtain the equation:

$$d\bar{æ} - \bar{\delta}_P \cdot d\bar{æ}/\bar{æ} = \bar{\delta}_\eta d\tau. \tag{4}$$

Here function $\bar{æ} = \bar{æ}\left(\bar{h}\right) = \bar{\delta}_P - \bar{\delta}_\zeta \bar{h}$, parameters

$\bar{\delta}_P = \sigma/r - \Delta p_c$, $\bar{\delta}_\zeta = \sigma(1 - \bar{\zeta}_K)/r$,$\bar{\delta}_\eta = (\sigma(1 - \bar{\zeta}_K)/h_K)^2/(3\eta)$.

Since for $\tau = 0$ the value $\bar{h} = 0$, $\bar{æ} = \bar{\delta}_P$, integrating (4) we find the dependence $F(\bar{h},\tau) = 0$:

$$\bar{h} + \bar{\sigma}_P \ln\left(1 - \bar{h}/\bar{\sigma}_P\right) = -\bar{\sigma}_\eta \tau, \tag{5}$$

where $\bar{\sigma}_P = \bar{\delta}_P/\bar{\delta}_\zeta = (\sigma - r\Delta p_c)/\left(\sigma(1-\bar{\zeta}_K)\right)$,$\bar{\sigma}_\eta = \bar{\delta}_\eta/\bar{\delta}_\zeta = r\sigma(1-\bar{\zeta}_K)/(3\eta h_K^2)$.

Hence, the average velocity of the meniscus movement in the capillary

$$\tilde{v}_M = \tilde{v}_M(\tau) = \left(\bar{\delta}_P r^2/3\eta h\right)\left(1 - \bar{h}/\bar{\sigma}_P\right). \tag{6}$$

If the ratio $\bar{h}/\bar{\sigma}_p \ll 1$, then from (5) and (6) we obtain the estimate of the functions $h(\tau)$, $\tilde{v}_M(\tau)$:

$$h \approx \left(2r(\sigma - r\Delta p_c)\tau/(3\eta)\right)^{1/2}, \tag{7}$$

$$\tilde{v}_M \approx \left(r(\sigma - r\Delta p_c)/6\eta\tau\right)^{1/2} \approx h/\left(\sqrt{2}\tau\right).$$

It is seen that immediately after the beginning of fluid movement in the capillary the remoteness of the meniscus increases proportionally to $\sqrt{\tau}$ at that velocity of its displacement decreases with time, the movement is slowed down. According to (6) the meniscus stops when $\bar{h} = \bar{\delta}_P$ i.e. at the distance from the movement start

$$h = h_{OK} = \frac{r\zeta_K(1-\Delta\bar{p}_c)}{\left(\Delta\bar{p}_c(1-\bar{\zeta}_K)\right)},$$

where $\Delta\bar{p}_c = r\Delta p_c/\sigma$– dimensionless pressure drop and $0 < \Delta\bar{p}_c < 1$ $0 < \Delta\bar{p}_c < 1$.

In accordance with (7) time to the meniscus stop will be approximately,

$$\tau = \tau_{OK} \approx 1.5 h_{OK}^2 \eta/\left(r\sigma(1-\Delta\bar{p}_c)\right). \tag{8}$$

From formula (8) it follows that the time to the meniscus stop is smaller the smaller are the fluid viscosity, the capillary width, the greater is the surface tension. The given regularities are quite consistent with the existing physical notions.

In the considered case the contact angle θ increases (see (3)) along the meniscus movement. However, the situation where on the contrary the angle θ decreases starting from θ_K is of great interest. In this case, we assume that dependence $\zeta = \zeta(\bar{h})$ is described by the relationship:

$$\zeta = \zeta(\bar{h}) = \zeta_K + (\bar{\zeta}_K - \zeta_K)\bar{h}.$$

By analogy with the previous case we introduce the auxiliary function $æ^* = æ^*(\bar{h}) = \delta_P^* + \delta_\zeta^* \bar{h}$, где $\delta_P^* = \zeta_K\sigma/r - \Delta p_c$, $\delta_\zeta^* = \sigma(\bar{\zeta}_K - \zeta_K)/r$, parameter $\delta_\eta^* = (\sigma(\bar{\zeta}_K - \zeta_K)/h_K)^2/(3\eta)$, using which we obtain the equation for finding $æ^*$:

$$dæ^* - \delta_P^* dæ^*/æ^* = \delta_\eta^* d\tau. \tag{9}$$

Taking into account that at the initial time $\tau = 0$ $(\bar{h} = 0)$ function $æ^* = \delta_P^*$, integrating (9), we determine:

$$\bar{h} - \sigma_P^* \ln\left(1 + \bar{h}/\sigma_P^*\right) = \sigma_\eta^* \tau,$$

where $\sigma_P^* = (\zeta_K - \Delta\bar{p}_c)/(\bar{\zeta}_K - \zeta_K)$,$\sigma_\eta^* = r\sigma(\bar{\zeta}_K - \zeta_K)/3\eta h_K^2$.

At that the average velocity of the meniscus movement

$$\tilde{v}_M = \tilde{v}_M(\tau) = \left(\frac{r^2\delta_P^*}{3\eta h}\right)\left(\frac{1+\bar{h}}{\sigma_P^*}\right),$$

approximately $\left(\bar{h}/\sigma_P^* \ll 1\right)$

$$h \approx \left(2r\sigma(\zeta_K - \Delta\bar{p}_c)\tau/3\eta\right)^{1/2}, \tag{10}$$

$$\tilde{v}_M \approx \left(r\sigma(\zeta_K - \Delta\bar{p}_c)/6\eta\tau\right)^{1/2} \approx h/\sqrt{2}\tau.$$

In this case, in contrast to the previous one when changing the variable ζ from ζ_K to $\bar{\zeta}_K\left(0 < \bar{h} < 1\right)$ the velocity of the meniscus movement is not equal to zero, the meniscus does not stop. After the lapse of control time τ_K from the beginning of the meniscus movement it will shift approximately at a distance h_K determined by the formula (10) and velocity of its movement is $\tilde{v}_K \approx h_K/\sqrt{2}\tau_K$. For fixed r, ζ_K, $\Delta\bar{p}_c$, time τ_K distance of the meniscus from the initial position increases with increasing of the surface tension σ and decreasing of the fluid viscosity η proportionally to $\sqrt{\sigma/\eta}$.

3. Conclusions

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In conclusion note that within the applied approach it is possible to take into account the impact on the dynamics of the meniscus movement in the capillary of the mass forces, change over time of the pressure drop of the media in contact, temperature.

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