

# Hierarchical stochastic model of terrain subsidence during tunnel excavation

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**Abstract.** In this contribution the Bayesian statistical method is applied to assess the expected probability distribution of the terrain subsidence in the course of tunnel excavation. The approach utilizes a number of simplifying assumptions regarding the system kinematics to arrive at a very simple model with just a few degrees of freedom. This deterministic model together with the intrinsic uncertainties of its parameters and measurement inaccuracies are used to formulate the stochastic model which defines a distribution of the predicted values of terrain subsidence. Assuming the measured data to be fixed, the stochastic model thus defines the likelihood function of the model parameters which is directly used for updating their prior distribution. This way the model parameters can be incrementally updated with each excavation step and the prediction of the model refined.

## 1. Introduction

Bayesian statistics and especially Bayesian parameter inference is a powerful tool for assessing uncertainty of complex systems. To understand its concepts we need to change our perception of probability. While in case of the observable quantity the probability density expresses how often the given value occurs (frequentist interpretation of probability), in case of unobserved quantity the probability density expresses how much we believe in that particular value (Bayesian interpretation of probability). Although the difference may seem subtle, it allows us to attribute the probability not only to measured data but also to model parameters.

Then, if there is a stochastic relationship between the unobserved quantities and observed data we can reappraise our prior knowledge about the unobserved quantities, i.e. their prior probability densities, and obtain a rationally updated knowledge – their posterior probability density.

The main advantage of this approach is its mathematical soundness: if our model (stochastic relations) is correct then the posterior belief is the most rational one given the prior belief and the observed data. On the other hand, one must be prepared that the integral statistics of posterior distribution such as the mean value, standard deviation or quantiles can be computed analytically only for a limited category of models. Posterior distribution obtained with models of an arbitrary structure has to be analyzed numerically via Markov chain Monte Carlo methods such as the Metropolis–Hasting algorithm, Gibbs sampling or the Hamiltonian Monte Carlo algorithm. Since these algorithms are implemented in several statistical programs we can concentrate only on the definition of the stochastic model and the prior distribution of its parameters.



This contribution presents a simple stochastic model including a simplified deterministic description of the evolution of settlement during tunnel construction as well as various input and measurement uncertainties. The parameters are updated based on data from monitoring in one tunnel cross-section. Then the settlement in the other cross-section is predicted both with and without taking the previously observed data into account.

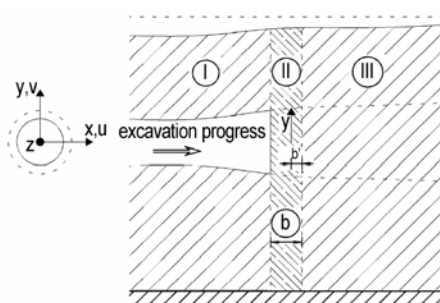
## 2. Deterministic model of sequential excavation of driven tunnel

In a typical engineering application the stochastic approach usually involves a deterministic model of the idealized reality. In our case the model predicts the evolution of displacements in the soil body induced by successive excavation of a driven tunnel. The model is founded on the assumption that the displacement field and the subsidence trough can be approximated with rather simple shape functions controlled by a small number of parameters. The complete formulation of the model goes beyond the scope of this paper so that only the basic model principles are provided. Further details can be found in [1].

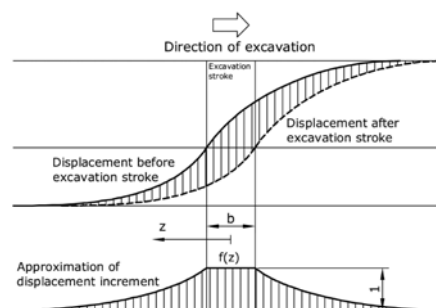
The model predicts the increment of vertical displacements following a single excavation stroke of length  $b$  as shown in Figure 1. Further we assume that the increment of displacements in segment II will vanish with the distance, i.e. the shape function  $f(z)$  decreases in segments I and III, see Figure 2. The distribution of the displacement increment in the  $x \times y$  plane is controlled by similar shape functions displaced in Figures 3, 4 and 5.

As has been shown in [1] the spacial distribution of the vertical displacement  $v(x, y, z)$  is completely defined by five model degrees of freedom (equivalent to nodal displacements in standard FEM formulation, see a considerably more complex approach taking also into account the interaction of upper and underground structure [2])  $V_1, V_2, V_3, \Delta V_1$  and  $\Delta V_2$ , see Figure 5, derived as a sum of an infinite series of their incremental values. The proposed kinematic relations then enable formulating the overall stiffness matrix and the load vector through the standard application of the principal of virtual work. The load vector due to the excavation forces is computed from the known geostatic stress.

As displayed in Figure 3, the model includes three layers of homogeneous material. If the real geological profile consists of more layers, the Young modulus in the particular model layer is computed as an average of the values in the profile sub-layers weighted by their thicknesses.

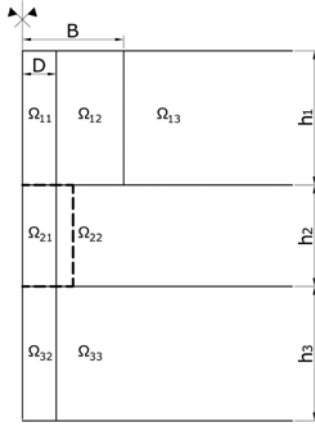


**Figure 1.** The model is divided to three segments in the longitudinal direction. The increment of the vertical displacement induced by the excavation of the segment II is computed at any point.

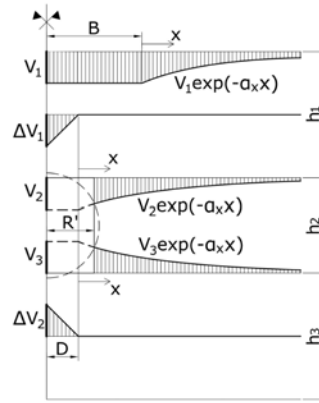


**Figure 2.** Shape function  $f(z)$  defines how the displacement increment evolves in the longitudinal direction. The rate of decrease of the function  $f(z)$  is controlled by parameter  $\alpha_z$ .

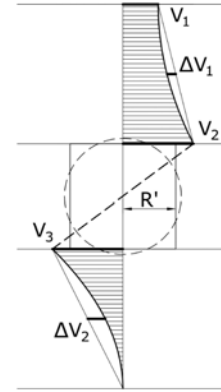
From the perspective of stochastic modeling, the deterministic model described above is abstracted as a black-box function taking the material parameters (Young modulus, Poisson ratio and density in each model layer), geometry (thicknesses of the model layers, the tunnel radius and the length of excavation step) and shape function parameters as an input and projecting them to the displacement



**Figure 3.** In  $x \times y$  plane the model is divided into several domains each governed by convenient shape function.



**Figure 4.** Shape functions in the  $x$  direction. The parameter  $\alpha_x$  defines the rate of decrease of the displacements.



**Figure 5.** The approximation in the vertical direction is controlled by auxiliary quantities  $\Delta V_1$  and  $\Delta V_2$ .

field  $v(\mathbf{x}) = v(x, y, z)$ , where  $\mathbf{x}$  is measured relatively to the position of the tunnel face, see Fig 1. For our particular case, this can be schematically written as

$$\left. \begin{array}{l} \text{Soil profile:} \quad \langle E_i, \nu_i, \rho_i, t_i \rangle, \quad i = 1, 2, 3 \\ \text{Geometry:} \quad h_1, h_2, h_3, R', b \\ \text{Shape parameters:} \quad \alpha_x, \alpha_z, D, B \end{array} \right\} \xrightarrow{\text{Model}} v(\mathbf{x}). \quad (1)$$

For a specific tunnel with a given geometry and a given soil profile some of the input parameters are known while the other might remain unknown or at least uncertain. The choice which parameters are known (constant) and which uncertain (assigned suitable prior distribution) is solely a matter of engineering judgment. In our practical example we limit our attention to five parameters only collected in vector  $\mathbf{b} = \{E_1, E_2, E_3, \alpha_x, \alpha_z\}^T$ . The remaining parameters are assumed to be known. The deterministic mapping function that projects the the vector of unknown variables on the vertical displacement then simplifies as above schema. For a

$$\mathbf{b} \xrightarrow{g_{\text{model}}} v(\mathbf{x}). \quad (2)$$

In such a case uncertainty of the displacement field is attributed to uncertainty of Young's moduli and the shape function parameters  $\alpha_x$  and  $\alpha_z$ . The details on how the model is used are given in Section 4.

### 3. Stochastic model of measured data

The above deterministic function is part of the stochastic model which takes uncertainty of the model parameters and the measurement error into account. According to the Bayes theorem the posterior joint probability distribution of the model parameters is proportional to the product of likelihood function and the prior joint probability distribution

$$p(\boldsymbol{\theta}|\mathbf{y}) \propto p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\theta}) \quad (3)$$

where vector  $\boldsymbol{\theta}$  denotes parameters of the stochastic model and  $\mathbf{y}$  represents the vector of observed data. Note, that the parameters  $\boldsymbol{\theta}$  are not directly observable, whereas the data  $\mathbf{y}$  can be measured. In fact, in the above relation the data  $\mathbf{y}$  are assumed to be already known and therefore fixed leaving both sides of Eq. (3) independent of  $\mathbf{y}$ .

Let's suppose that the vertical displacement in the vicinity of the tunnel tube will be measured at  $n$  points with known coordinates  $(x_j, y_j, z_j)$  where  $j = 1 \dots n$ . We denote the measured values as  $v_j$  and express our belief about probability distributions by the following set of relations

$$v_j \sim \mathcal{N}(\bar{v}_j, \sigma_v), \quad (4)$$

$$\bar{v}_j = g_{\text{model}}(\mathbf{b}(\mathbf{x}_j)) \quad (5)$$

$$E_i \sim \mathcal{U}(E_{i,\min}, E_{i,\max}), \quad i = 1, 2, 3 \quad (6)$$

$$\alpha_x \sim \mathcal{U}(\alpha_{x,\min}, \alpha_{x,\max}), \quad (7)$$

$$\alpha_z \sim \mathcal{U}(\alpha_{z,\min}, \alpha_{z,\max}), \quad (8)$$

$$\sigma_v \sim \mathcal{U}(\sigma_{v,\min}, \sigma_{v,\max}). \quad (9)$$

Relation (4) implies that the measured displacements  $v_j$  are normally distributed around the theoretical mean value  $\bar{v}_j$  with standard deviation  $\sigma_v$ . This way the measurement error and the modeling uncertainty are jointly covered by a simple stochastic relation. The theoretical mean values  $\bar{v}_j$  are a direct outcome of the deterministic model  $g_{\text{model}}$  as prescribed by (5). These two relations form the model likelihood function. Finally, relations (6)–(9) define the prior probability distributions of the top-level model parameters. In particular, the parameters  $E_1, E_2, E_3, \alpha_x, \alpha_z$  and  $\sigma_v$  are described by uniform prior distributions with specified limits.

#### 4. Example and results

Two cross-sections of the road tunnel Blanka of average radius  $R = 5$  m constructed in Prague are chosen to provide real-world data for the stochastic model. Before the construction of the tunnel a borehole was drilled in each monitored section. The soil profile has been documented and the borehole was equipped with extensometers to allow for monitoring of the vertical displacement developed during the tunnel construction. A simplified stratigraphy of the two monitored sections denoted as J22 and J24, 101 m apart, is provided in Tables 1 and 2. The ranges of the Young modulus were adopted from the ČSN 73 1001 standard for each of the soil class. The stroke size  $b$  was set equal to 2 m. The remaining model parameters equal to  $D = 4$  m and  $B = 10$  m were derived from an identification procedure outlined in [1].

The displacement evolution monitored only at extensometer J22 on terrain and just above the tunnel crown, i.e. at a depth of 15 m, were used in the inference procedure. In particular, the measurements of vertical displacements from seven different positions of the tunnel face relative to the monitored section were adopted. As shown in Figure 6 these were taken from the interval -82 m where the tunnel face has not yet reached the monitored profile to 85 m where the tunnel face has already passed the extensometer. The total number of measurements in Eq. (5) thus amounted to  $n = 14$ .

The ranges of the uniform prior distribution for the Young moduli in the soil layers were taken from Table 1 and the parameters  $\alpha_x$  and  $\alpha_z$  were assumed to be uniformly distributed between 0.01 and 1. The prior distribution for the standard deviation of the measurement error  $\sigma_v$  was also uniform with ranges from 0.1 to 5 mm.

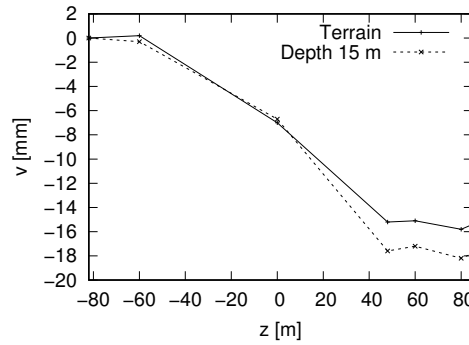
**Table 1.** Soil profile in borehole (J22).

i	Depth [m]	Soil type	$E_i$ [MPa]	$\nu_i$ [–]
1	0–4.6	Clay (F3)	3–15	0.35
2	4.6–7	Gravel (G3)	80–100	0.25
3	7–20	Slate (R3)	60–300	0.2

**Table 2.** Soil profile in the borehole (J24).

i	Depth [m]	Soil type	$E_i$ [MPa]	$\nu_i$ [–]
1	0–5.5	Clay (F3)	3–15	0.35
2	5.5–8.8	Gravel (G3)	80–100	0.25
3	8.8–18.5	Slate (R4)	40–120	0.25

The hierarchical model given by equations (4)–(9) was specified in a dialect of the BUGS language [3] and run in JAGS program [4]. Due to the limited range of operators and functions available in the JAGS

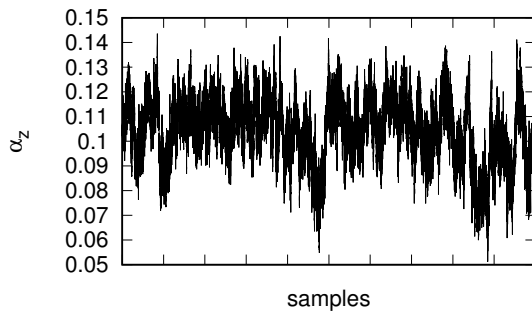


**Figure 6.** Development of the measured vertical displacement  $v$  on terrain and at the depth of 15 m with the position of the tunnel face  $z$ .

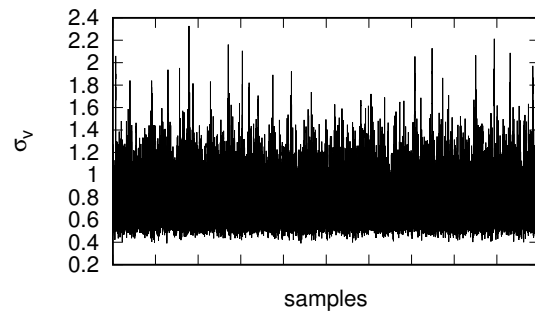
program the formulation of the deterministic function  $g_{\text{model}}$  needs to be approximated. The Taylor series in the form

$$\begin{aligned} \bar{v}_j &= g_j(\mathbf{b}(\mathbf{x}_j)) \approx g_j(\bar{\mathbf{b}}) + \sum_{r=1}^5 \frac{\partial g_j}{\partial b_r} \bigg|_{\mathbf{b}=\bar{\mathbf{b}}} (b_r - \bar{b}_r) \\ &+ \frac{1}{2} \sum_{r=1}^5 \sum_{s=1}^n \frac{\partial^2 g_j}{\partial b_r \partial b_s} \bigg|_{\mathbf{b}=\bar{\mathbf{b}}} (b_r - \bar{b}_r)(b_s - \bar{b}_s) + \dots \end{aligned} \quad (10)$$

is a suitable tool for such approach. In what follows, the model is further reduced to a constant and the linear term approximating  $g_{\text{model}}(E_1, E_2, E_3, \alpha_x, \alpha_z)$  at  $\bar{E}_1 = 8$  MPa,  $\bar{E}_2 = 90$  MPa,  $\bar{E}_3 = 120$  MPa,  $\bar{\alpha}_x = 0.1$  and  $\bar{\alpha}_z = 0.1$ .



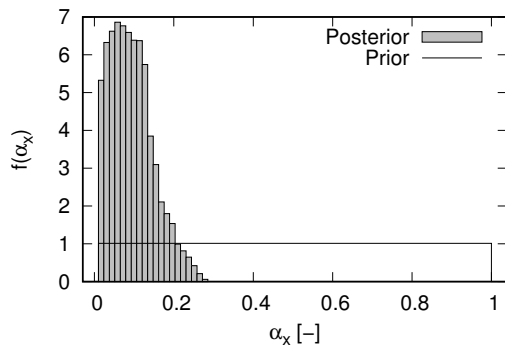
**Figure 7.** Markov chain of the generated samples of the parameter  $\alpha_z$ .



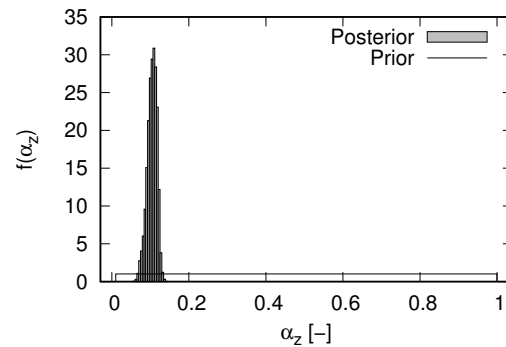
**Figure 8.** Markov chain of the generated samples of the parameter  $\sigma_v$ .

The program uses Gibbs sampling to generate the samples of the model parameters according to the joint posterior distribution. The burn-in period of 1000 samples, which were discarded, was followed by generating two chains of 50000 samples. For illustration, the Markov chains for parameters  $\alpha_z$  and  $\sigma_v$  are displayed in Figures 7 and 8. Both chains show sufficient mixing of the generated samples indicating proper settings of the Gibbs sampling algorithm. Therefore, the generated histograms should converge to the theoretical continuous distributions.

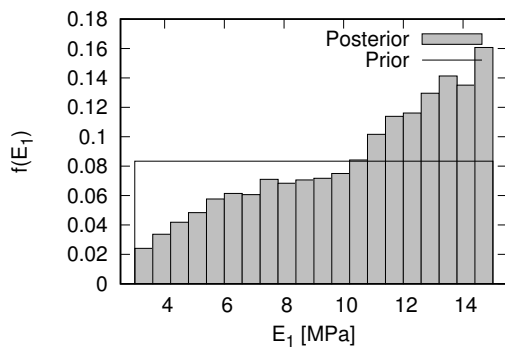
The histograms computed from the generated Markov chains of the top-level model parameters are displayed in Figures 9–14 along with their prior uniform distributions. The differences of the prior (uniform) distributions and the posterior distributions (visualized in form of histograms) tell us how the



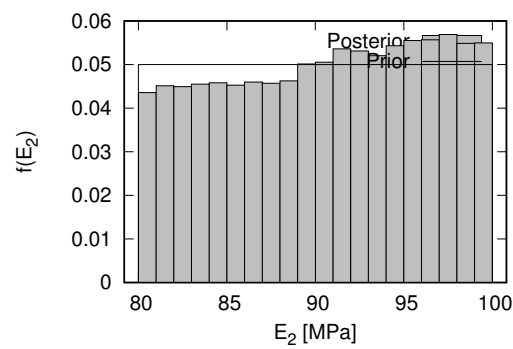
**Figure 9.** Prior and posterior distribution of parameter  $\alpha_x$ .



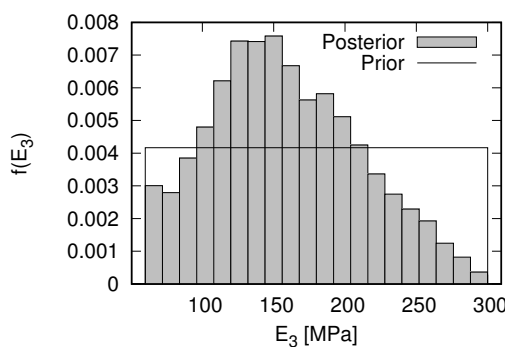
**Figure 10.** Prior and posterior distribution of parameter  $\alpha_z$ .



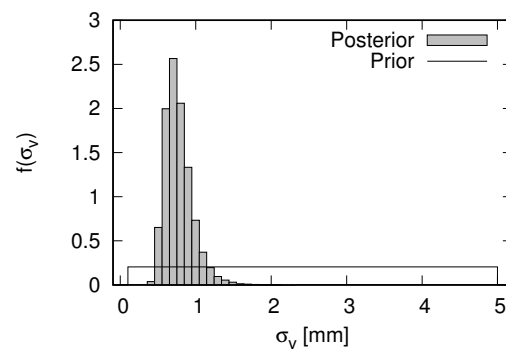
**Figure 11.** Prior and posterior distribution of the Young modulus of clay layer  $E_1$ .



**Figure 12.** Prior and posterior distribution of the Young modulus of gravel layer  $E_2$ .



**Figure 13.** Prior and posterior distribution of the Young modulus of rock layer  $E_3$ .



**Figure 14.** Prior and posterior distribution of the standard deviation of measurement error of vertical settlement  $\sigma_v$ .

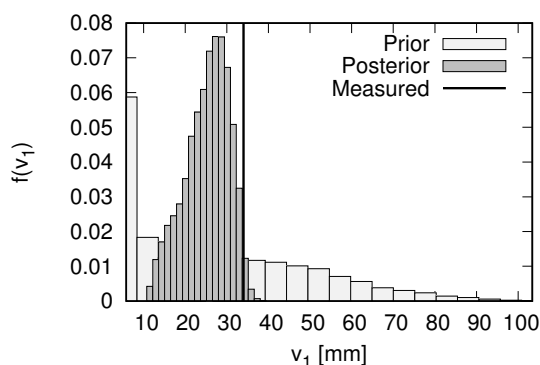
prior knowledge about the value of the parameter changed after the data were observed. These graphs suggest some partial conclusions. Clearly, the algorithm managed to identify the shape parameters  $\alpha_x, \alpha_z$ , Figures 9 and 10, together with the measurement error in Figure 14 relatively well. It also appears, perhaps even intuitively expected, that the direct impact of the gradually excavated mid section on the settlement measured at the selected points is not significant. Additionally, the results plotted in Figure 14 may draw one to the conclusion that the original soil classification might be erroneous.

This also fully uncovers the advantage of the Bayesian inference when adopted full scale analysis incorporating in-situ measurements rather than just measurements from small scale experiments.

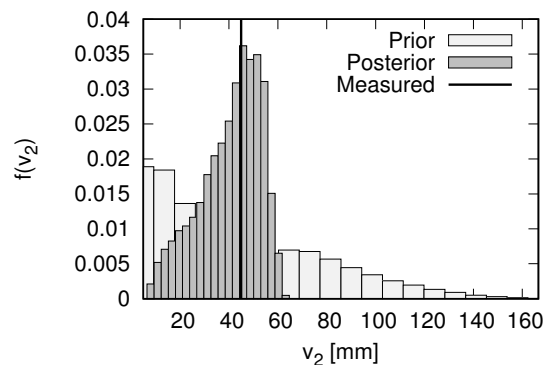
The principal objective of the application of the Bayesian inference is to improve original predictions given new (updated) information about the model parameters. In this paper, this is demonstrated by providing predictions of the settlements measured at a different section than was the one exploited in the updating process. The predicted settlements were found assuming:

- (i) A vague knowledge of model parameters specified by their prior distribution with no knowledge of their potential statistical dependence.
- (ii) Improved (updated) knowledge of model parameters represented by their joint probability distribution function available from the already generated Markov chain.

In the first case variability of the settlement was predicted using simple Monte Carlo method with individual outcomes of vector  $\mathbf{b}$  sampled from previously adopted statistically independent uniform prior distributions. In the second case, these outcomes were adopted directly from the Markov chains of parameters obtained within the inference procedure. More specifically, only the updated parameters  $E_1$ ,  $E_2$ ,  $\alpha_x$ ,  $\alpha_z$  and  $\sigma_v$  were used in the prediction. The updated distribution of the parameter  $E_3$  was not used here as it represented the Young modulus of the rock layer of class R3 in section J22 while the rock layer in J24 section is classified as R4, i.e. more fractured. Due to this subtlety, a separate prior uniform distribution with the range 40–120 MPa was considered for the Young modulus of the rock layer in section J24. The range was again taken from the ČSN 73 1001 standard. Note that the measured settlements from this cross-section were not used within the updating procedure.



**Figure 15.** Prior predictive and posterior predictive distribution of the final vertical displacement at extensometer J24 at the terrain.



**Figure 16.** Prior predictive and posterior predictive distribution of the final vertical displacement at extensometer J24 at a depth of 15 m.

The settlements derived from prior distributions (Prior) and the posterior joint probability distribution (Posterior) in section J24 are shown in Figures 15 and 16. They correspond to the tunnel face being at a distance of 74 m passed section J24. Apart from the distributions the figures show also the values of the actual measured vertical displacements. The 98% confidence interval computed from the posterior predictive distribution (approach ii) of the total settlement on the terrain is 12.6 mm – 34.6 mm, and 10.3 mm – 60.1 mm for the displacement at a depth of 15 m, respectively. This is a more useful piece of information when compared to the significantly wider confidence interval for prior predictive distributions (approach i) that are 4.4 mm – 85.1 mm for terrain, and 1.9 mm – 134.4 mm for the displacement at 15 m depth, respectively. Both measured values, i.e. 34 mm at the terrain, and 45 mm at the depth of 15 m, respectively, fit into the prior as well as the posterior predicted confidence intervals.



## 5. Conclusions

The presented analysis demonstrated that the originally quite wide range of the admissible values of the model parameters is significantly narrowed when the observed data are taken into account. Moreover, the updating procedure reflects the structure of the stochastic model. Due to the model structure some of the parameters are refined substantially while the distribution of the other parameters may remain practically unchanged. This is natural because some of the parameters do not influence the type of the observed data and thus the knowledge of the data does not refine their prior distribution. The updated set of parameters was used in the stochastic model for the prediction of the yet unobserved data, i.e. the settlements evolution in a particular tunnel cross-section. Recall that the purpose of the stochastic model is not to predict the settlements with a maximum accuracy but to predict its probability distribution. We are generally interested in the distribution of the predicted data, i.e. their confidence interval and not the single most probable value.

Nevertheless, the predictions obtained by the stochastic model outlined in this paper can be further improved in several ways. First and foremost, the prediction are improved with additional observed data. This does not only mean to monitor more sections but also to collect more types of the data such as the displacement sideways to the tunnel tube.

Additionally, the deterministic model used within the stochastic description can itself be improved. As for the reduced model for successive excavation, the introduction of the nonlinear material would make the analysis more realistic and allowing us to take the soil strength into account. Finally, the approximation of the model by means of the Taylor series (10) could be expanded by adding the higher order terms.

For reference, a similar formulation based on the finite element method and a nonlinear material is formulated in [5]. When comparing these two approaches, the following peculiarities deserve mentioning:

- (i) The effectiveness of the newly proposed approach exploiting the simplified deterministic model for the prediction of the surface settlement.
- (ii) The robustness of the Gibbs sampling algorithm allowing us to reliably capture the statistical dependence of the input parameters (components of vector  $\mathbf{b}$ ).

## Acknowledgments

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