

A preconditioner for symmetric saddle point matrices

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Abstract. In this paper, a new preconditioner for numerical solutions of symmetric indefinite linear systems is presented. The new preconditioner called as product preconditioner is constructed through the product of two fairly simple preconditioners. The eigenvalues distribution and the form of the eigenvectors of the product preconditioned matrix are analyzed. Numerical experiments illustrate the effectiveness of product preconditioner.

1 Introduction

Our aim is to find the numerical solutions of two-by-two block indefinite linear systems of the form

$$Au = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite (SPD), $B \in \mathbb{R}^{n \times m}$ has full rank and $n > m$. Under these assumptions, the coefficient matrix A is indefinite and nonsingular [1]. This system has a wide application, e.g., the finite element and mixed finite element methods for solving the Navier-Stokes equations, constrained and generalized least squares problems, constrained optimization and fluid dynamics [2--6].

In recent years, there are plenty of papers of preconditioners for large sparse linear systems in saddle point problem, see [7-19]. Murphy, Golub and Wathen [8] proposed a block diagonal Schur complement preconditioner. This preconditioner has been improved by using of a splitting of the (1,1) block A [9]. Furthermore, Siefert and Sturler [11] extended the results of [9] to matrices with a nonzero (2,2) block. Keller, Gould and Wathen [12] investigated a class of constraint preconditioners for the indefinite linear system (1). Moreover, the eigenvalue distribution and the form of the eigenvectors of the preconditioned matrix as well as an upper bound on the degree of its minimal polynomial were all discussed. Dollar [16] introduced a new factorization for the constraint preconditioner and extended this idea by allowing the (2,2) block to be symmetric and positive semidefinite [17]. In addition, Cao [18, 19] extended the constraint preconditioner to nonsymmetric and generalized saddle point problems. More preconditioners for saddle point problems can be found in the comprehensive paper [1].

The paper is organized as follows. In Section 2, we briefly introduce background on stationary iterations and matrix splittings. In Section 3, we analyze the eigenvalues distribution and the form of the eigen-vectors of the product preconditioned matrix. Numerical experiments are presented in Section 4. Finally, conclusions and future work are discussed in Section 5.

2 Background and product preconditioner



We borrow the background material on stationary iterations and matrix splittings in [13, 20] to construct the proposed product preconditioner in this section.

2.1 Stationary iterations and matrix splittings

Consider $Au = b$, is a large sparse linear system of the form where A is a square and nonsingular matrix and b is given. Stationary iterative methods may be attractive by using a splitting of the coefficient matrix A as $A = M - N$, where M is a nonsingular matrix. In [20], Benzi and Szyld defined a related approach by the alternating iteration

$$\begin{cases} u^{k+1/2} = M_1^{-1}N_1u^k + M_1^{-1}b \\ u^{k+1} = M_2^{-1}N_2u^{k+1/2} + M_2^{-1}b \end{cases} \quad (2)$$

where $A = M_1 - N_1 = M_2 - N_2$ is the splitting of A and u_0 is an initial vector. Results about the convergence of the alternating iteration are given in [20]. In addition, by making use of the nonsingular matrices M_1 and M_2 , Benzi and Szyld constructed a splitting $A = M - N$ where

$$M^{-1} = M_2^{-1}(M_1 + M_2 - A)M_1^{-1}$$

2.2 Product preconditioner

The product preconditioner is constructed through the multiplication of two fairly simple preconditioners. The first preconditioner is the block diagonal preconditioner

$$M_{bd} = \begin{pmatrix} A & O \\ O & B^T A^{-1} B \end{pmatrix} \quad (3)$$

Note that M_{bd} is nonsingular since A is a symmetric and positive matrix and B is full rank.

The second preconditioner is the nonsingular constraint preconditioner

$$M_{sc} = \begin{pmatrix} G & B \\ B^T & O \end{pmatrix}$$

presented by Keller, Gould and Wathen in [12] where $G \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite approximation of A .

The product preconditioner M_{ps}^{-1} is given

$$M_{ps}^{-1} = M_{sc}^{-1}(M_{bd} + M_{sc} - A)M_{bd}^{-1} \quad (4)$$

where the matrix

$$M_{bd} + M_{sc} - A = \begin{pmatrix} G & O \\ O & B^T A^{-1} B \end{pmatrix}$$

is invertible. Hence, M_{ps}^{-1} is well-defined. From (4), we obtain

$$\begin{aligned} M_{ps} &= M_{bd}(M_{bd} + M_{sc} - A)^{-1}M_{sc} \\ &= \begin{pmatrix} A & O \\ O & B^T A^{-1} B \end{pmatrix} \begin{pmatrix} G^{-1} & O \\ O & (B^T A^{-1} B)^{-1} \end{pmatrix} \begin{pmatrix} G & B \\ B^T & O \end{pmatrix} \\ &= \begin{pmatrix} A & AG^{-1}B \\ B^T & O \end{pmatrix} \end{aligned} \quad (5)$$

such that

$$A - M_{ps} = \begin{pmatrix} O & B - AG^{-1}B \\ O & O \end{pmatrix}$$

and

$$M_{ps}^{-1}A = M_{ps}^{-1}(A - M_{ps})^{-1}$$

The matrix M_{ps} can be factorized as

$$M_{ps} = \begin{pmatrix} A & A^T G^{-1} B \\ B^T & O \end{pmatrix} \begin{pmatrix} I & O \\ B^T A^{-1} & I \end{pmatrix} \begin{pmatrix} A & A^T G^{-1} B \\ O & -B^T G^{-1} B \end{pmatrix}$$

then

$$\begin{aligned} M_{ps}^{-1} &= \begin{pmatrix} A & A^T G^{-1} B \\ O & -B^T G^{-1} B \end{pmatrix}^{-1} \begin{pmatrix} I & O \\ B^T A^{-1} & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} A^{-1} - G^{-1} B (B^T G^{-1} B)^{-1} B^T A^{-1} & G^{-1} B (B^T G^{-1} B)^{-1} \\ (B^T G^{-1} B)^{-1} B^T A^{-1} & -(B^T G^{-1} B)^{-1} \end{pmatrix} \end{aligned}$$

The product preconditioned matrix $M_{ps}^{-1}A$ can be expressed as

$$M_{ps}^{-1}A = \begin{pmatrix} I & A^{-1}B - G^{-1}B(B^T G^{-1}B)^{-1}B^T A^{-1}B \\ O & (B^T G^{-1}B)^{-1}B^T A^{-1}B \end{pmatrix} \quad (6)$$

3 Numerical experiments

In this section, we report on numerical results to illustrate the effectiveness of the product preconditioner (PS). The block diagonal preconditioner (BD) is

$$M_{bd} = \begin{pmatrix} G & O \\ O & B^T A^{-1} B \end{pmatrix} \quad (7)$$

presented in [8, 9, 11]. The block triangular preconditioner (BT) is

$$M_{bt} = \begin{pmatrix} G & B \\ O & B^T A^{-1} B \end{pmatrix} \quad (8)$$

considered in [8, 11].

There are various strategies to choose G in the above mentioned PS, BD and BT. Here, we take G to be the tridiagonal matrix of the (1,1) block matrix A , i.e., $G = \text{tridiag}(A)$. In addition, we set G as the incomplete Cholesky factorizations of A ,

$$R_1 = \text{cholinc}(A, 0), G_1 = R_1^T R_1 \quad (9)$$

and

$$R_2 = \text{lunic}(A, 0.01), G_2 = R_2^T R_2 \quad (10)$$

where $R_1 \in \mathbb{R}^{n \times n}$ and $R_2 \in \mathbb{R}^{n \times n}$ are the upper triangular matrices, ‘0’ in (9) denotes there is no fill-in in the process of the incomplete Cholesky factorization of A and 0.01 in (10) is the value of the drop tolerance of the incomplete Cholesky factorization of A .

Here the test problem that has been considered in [8] is used in our numerical experiments. That is, in the two-by-two block indefinite linear system (1), the matrix

$$\begin{aligned} A &= \begin{pmatrix} I \otimes T + T \otimes I & O \\ O & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2} \\ B &= \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2} \end{aligned}$$

and

$T = \frac{1}{h^2} \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p \times p}$, $F = \frac{1}{h^2} \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{p \times p}$ with \otimes being the Kronecker product symbol and $h = 1/(1+p)$ the discretization mesh size. Let $n = 2p^2$ and $m = p^2$, then the total number of variables is $m+n = 3p^2$. In our experiments, we consider the following four cases when $p = 8$, $p = 16$, $p = 24$ and $p = 32$ and choose $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ such that the exact solution of (1) $(x^T, y^T)^T = (1, 1, \dots, 1)^T \in \mathbb{R}^{n+m}$.

GMRES method [30] with the right preconditioning is used as a representative iterative solver in our experiments. In particular, we let GMRES method with at most 10 restarts be used in our experiments. All the iterations are started from an $(n+m) \times 1$ zero vector and terminated when $\text{RES} = \|b - Au\|_2 / \|b\|_2 \leq 10^{-9}$. Finally, numerical results for different preconditioners are reported in the following tables where IT denotes the number of iterations.

Table 1 gives numerical results of the BD, BT and PS preconditioned GMRES methods for this test problem when $G = \text{tridiag}(A)$. As we have seen from Table 1, the iteration counts and RES of the PS preconditioned GMRES method are less than those of the BD and BT preconditioned GMRES methods. For the BD preconditioned GMRES method, iteration counts are reduced by around 80% and for the BT preconditioned GMRES method, iteration counts are reduced by around 78%.

Table 2 provides numerical results of the BD, BT and PS preconditioned GMRES methods for this test problem when G is taken to be the incomplete Cholesky factorization of A , i.e., $G = G_1$ given in (9). It is clear that the PS preconditioned GMRES method seems to be the most efficient. Iteration counts of the BD and BT preconditioned GMRES methods are reduced by around 76% and 70%, respectively.

Table 3 supplies numerical results of the BD, BT and PS preconditioned GMRES methods for this test problem when G is taken to be the incomplete Cholesky factorization of A , i.e., $G = G_2$ given in (10). From Table 3, we also find that the iteration counts and RES of the PS preconditioned GMRES method are less than those of the BD and BT preconditioned GMRES methods. For the BD preconditioned GMRES method, iteration counts are reduced by around 72% and for the BT preconditioned GMRES method, iteration counts are reduced by around 65%.

Table 1: Numerical results of the BD, BT and PS preconditioned GMRES methods for this test problem when $G = \text{tridiag}(A)$.

p		16	20	24	28
BD	IT	239	360	700	699
	RES	9.99e-10	8.10e-10	9.30e-10	8.95e-10
	TIME	0.8891	2.5288	8.3355	11.3487
BT	IT	65	90	108	139
	RES	9.87e-10	9.74e-10	9.39e-10	1.32e-05
	TIME	1.1883	3.3358	7.9287	30.9748
PS	IT	20	38	53	68
	RES	6.18e-10	9.96e-10	9.26e-10	7.14e-10
	TIME	0.0512	0.1408	0.2639	0.3687

Table 2: Numerical results of the BD, BT and PS preconditioned GMRES methods for this test problem when $G = G_1$ given in (9).

p		16	20	24	28
BD	IT	96	102	109	139
	RES	5.19e-10	9.14e-10	9.52e-10	9.45e-10
	TIME	1.2761	4.5724	7.9728	23.5386
BT	IT	86	99	115	137
	RES	5.18e-10	7.46e-10	9.65e-10	9.79e-10
	TIME	2.0907	5.8672	15.5421	45.8418
PS	IT	36	52	57	60
	RES	8.81e-10	9.60e-10	5.11e-10	7.91e-10

	TIME	0.0671	0.2316	0.5030	1.0124
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Table 3: Numerical results of the BD, BT and PS preconditioned GMRES methods for this test problem when $G = G_2$ given in (10).

p		16	20	24	28
BD	IT	45	52	58	62
	RES	7.50e-10	9.90e-10	4.94e-10	9.01e-10
	TIME	0.7084	2.2211	4.4371	10.6812
BT	IT	43	49	57	59
	RES	7.53e-10	5.07e-10	2.70e-10	7.48e-10
	TIME	1.1578	3.0107	7.3526	22.7883
PS	IT	22	25	28	30
	RES	9.12e-10	7.05e-10	6.85e-10	6.02e-10
	TIME	0.0382	0.1131	0.2432	0.5126

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