

The study on the identity of property under operation on binary relation

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Abstract. This paper is to study the links between basic properties and several operations of the binary relations, this means what will happen to original property under binary relation's operations, such as converse, composition, union, intersection, relative complement(difference) and closure operations, whether the original properties is maintained or not? Also, it can be called the identity of properties under operations on binary relations.

1 Introduction

The concepts, contents, thoughts and methods of relation in set theory are abundant and basic, we know the relation is also a set, so broadly speaking, the concept of set have important significance for mathematics, computer and other areas. It is known as the basis of mathematics, as well for accurate description of advanced concepts and mathematical reasoning^[4]. At the end of nineteenth century, G. Cantor (1845-1918), a mathematician who was born in St Petersburg, Russia, is of German origin, laid a foundation for the set theory. Especially, he made a deep research on infinite sets, put forward some new concepts that have a large impact on many previous ideas and traditional ideas, and thus he suffered some sharp criticism^[3,16]. Furthermore, he founded a simple and intuitive set theory (naive set theory), which has some antinomies^[17], the solution is the axiomatization of set theory.

The contents of axiomatic set theory is very profound, in which the axiom of choice^[7] is still an argument, even the infinite set of concepts, such as cardinal number^[6], ordinal number^[15], and so on, are also not easy to understand. As time and learning schedule is urgent, this paper is not to discuss them, here we only talk about binary relations. In many relationships, the relationship between the two things is the most basic, but it has many types, properties, operations, there are also a lot of theoretical results and practical applications with it. For example, equivalence relation, partial ordering relation are all often contacted, like the congruence is an equivalence relation^[11], and the number set is a poset. The different relation types are usually the combination of different properties^[9], similarly, operations of relationship is also rich in content, such as closure operations, there is a very wide range of applications. The main content of this paper is to discuss the impact of several operations (inverse, composition, union, intersection, relative complement (difference) and closure operations) on the original basic properties (reflexive, irreflexive, symmetric, antisymmetric and transitive), whether can remains the same or not? What will happen to it? The concrete results can be represented by a two-dimensional table^[8], as the following Table 1, the $\sqrt{}$ express "can keep", the meaning of \times has a little more complicated, can generally be said "don't keep", but it sometimes may be divided into various situations. Details will be discussed in the following proofs, when we try to judge the property of a relation, which can be immediately proved from definition, in which case, we should make the most of propositional logic or predicate logic calculus to prove it. Of course, we can also use the equivalence



condition of the relational property to prove it with set expressions indirectly. In this article, we will give the proof separately from these two points.

Table 1: links between basic properties and several operations of the binary relations

Original proper -ty Operation	reflexive	Irreflex-ive	Symm-etric	Antisym-metric	Transit-ive
R_1^{-1}	✓	✓	✓	✓	✓
$R_1 \cup R_2$	✓	✓	✓	×	×
$R_1 \cap R_2$	✓	✓	✓	✓	✓
$R_1 - R_2$	×	✓	✓	✓	×
$R_1 \circ R_2$	✓	×	×	×	×
$r(R_1)$	✓	×	✓	✓	✓
$s(R_1)$	✓	✓	✓	×	×
$t(R_1)$	✓	×	✓	×	✓

2 Preparatory knowledge

Definition^[2,10] **1** Ordered pairs $\langle x, y \rangle$ is defined to be $\{\{x\}, \{x, y\}\}$. This definition was given by Kazimierz Kuratowski (he is a Pole) in 1921, and is the definition in general use today.

Definition 2 Suppose that we have two sets A and B, and we form ordered pairs $\langle x, y \rangle$ with $x \in A$ and $y \in B$. The collection of all such pairs is called the Cartesian product $A \times B$ of A and B:

$$A \times B = \{\langle x, y \rangle \mid x \in A \wedge y \in B\}.$$

Definition^[5] **3** A binary relation is a set of ordered pairs, denoted as R, we sometimes write xRy in place of $\langle x, y \rangle \in R$. If A and B are sets and $R \subseteq A \times B$, we call R a binary relation between A and B, a relation $R \subseteq A \times A$ is called a binary relation on A. The universal relation on A is

$$E_A = \{\langle x, y \rangle \mid x \in A \wedge y \in A\} = A \times A$$

The identity relation on A is

$$I_A = \{\langle x, x \rangle \mid x \in A\}$$

Definition^[5] **4** If R is a binary relation, its inverse operation is

$$R^{-1} = \{\langle x, y \rangle \mid yRx\}$$

Definition^[5] **5** If F, G are all binary relation, their composition operation is

$$F \circ G = \{\langle x, y \rangle \mid \exists t (\langle x, t \rangle \in F \wedge \langle t, y \rangle \in G)\}$$

Definition^[5] **6** If R is a binary relation on set A, n is a natural number, the powers R^n is defined below:

a) $R^0 = \{\langle x, x \rangle \mid x \in A\} = I_A$; b) $R^n = R^{n-1} \circ R$ ($n \geq 1$)

There are five main properties of the relationship: reflexivity, irreflexivity, symmetry, anti-symmetry, transitivity. Defined as follows:

Let R be a binary relation on A,

a) If $\forall x (x \in A \rightarrow \langle x, x \rangle \in R)$ is true, then R is reflexive on A;

b) If $\forall x (x \in A \rightarrow \langle x, x \rangle \notin R)$ is true, then R is irreflexive on A;

c) R is symmetric on A

$$\Leftrightarrow \forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \rightarrow \langle y, x \rangle \in R)$$

d) R is anti-symmetric on A

$$\Leftrightarrow \forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \wedge \langle y, x \rangle \in R \rightarrow x = y)$$

$$\Leftrightarrow \forall x \forall y (x, y \in A \wedge \langle x, y \rangle \in R \wedge x \neq y \rightarrow \langle y, x \rangle \notin R)$$

e) R is transitive on A

$$\Leftrightarrow \forall x \forall y \forall z (x, y, z \in A \wedge \langle x, y \rangle \in R \wedge \langle y, z \rangle \in R$$

$$\rightarrow \langle x, z \rangle \in R)$$

Definition 7 Let R and R' be all relation on nonempty set A , $R \subseteq R'$ is known, if R' satisfy: a) is reflexive (symmetric or transitive); b) for any reflexive (symmetric or transitive) relation R'' , if $R \subseteq R''$, then $R' \subseteq R''$, we say that this new relation R' is called the reflexive (symmetric or transitive) closure of R , respectively, they are separately denoted $r(R)$, $s(R)$, $t(R)$, $t(R)$ is usually denoted R^+ , $tr(R)$ is usually denoted R^* .

Here are a few main theorems,

Theorem 1 Let R be a binary relation on nonempty set A , we can get some results below:

a) R is reflexive $\Leftrightarrow I_A \subseteq R$; b) R is irreflexive $\Leftrightarrow I_A \cap R = \emptyset$;

c) R is symmetric $\Leftrightarrow R = R^{-1}$; d) R is antisymmetric

$\Leftrightarrow R \cap R^{-1} \subseteq I_A$; e) R is transitive $\Leftrightarrow R \circ R \subseteq R$.

Theorem 2 Let R be a binary relation on nonempty set A , has:

a) R is reflexive if and only if $r(R) = R$; b) R is symmetric if and only if $s(R) = R$; c) R is transitive if and only if $t(R) = R$.

Theorem 3 Let R be a binary relation on nonempty set A , has:

a) $r(R) = R \cup R^0$; b) $s(R) = R \cup R^{-1}$; c) $t(R) = R \cup R^2 \cup R^3 \cup \dots$.

Theorem 4 Let R be a binary relation on nonempty set A and

$R_1 \subseteq R_2$, then:

a) $r(R_1) \subseteq r(R_2)$; b) $s(R_1) \subseteq s(R_2)$; c) $t(R_1) \subseteq t(R_2)$.

The detailed proof of the above theorems needs no further elaboration, please refer to the relevant literature. Now, we begin to prove the results of Table 1.

3 Proof of concrete results

3.1 Judgment of the properties of the inverse operation

a) If R_1 is reflexive on A , then R_1^{-1} is also reflexive.

Proof 1 (by definition)^[13]. x is chosen arbitrarily in A ,

$$x \in A \Rightarrow \langle x, x \rangle \in R_1 \Rightarrow \langle x, x \rangle \in R_1^{-1}$$

this accord with the definition of reflexivity, so a) is true;

Proof 2 (from the set expression of the theorem 1 in the section 2). Because $I_A \subseteq R_1$, so $I_A^{-1} \subseteq R_1^{-1}$, also because $I_A = I_A^{-1}$, so $I_A \subseteq R_1^{-1}$, therefore a) is true by Theorem 1.a).

b) If R_1 is irreflexive on A , then R_1^{-1} is also irreflexive.

Proof 1. By definition, it is similar to the above, omit it.

Proof 2. We have $I_A \cap R_1 = \emptyset$ from R_1 is irreflexive, then

$$I_A \cap R_1^{-1} = I_A^{-1} \cap R_1^{-1} = (I_A \cap R_1)^{-1} = \emptyset^{-1} = \emptyset$$

so b) is true by Theorem 1.b).

c) If R_1 is symmetric on A , then R_1^{-1} is also symmetric.

Proof 1. For every $\langle x, y \rangle \in A$, we shall see that

$$\langle x, y \rangle \in R_1^{-1} \Rightarrow \langle y, x \rangle \in R_1 \Rightarrow \langle x, y \rangle \in R_1 \Rightarrow \langle y, x \rangle \in R_1^{-1}$$

Proof 2. Because $R_1 = R_1^{-1}$, so $(R_1^{-1})^{-1} = R_1 = R_1^{-1}$, it is thus evident that proposition c) is true.

d) If R_1 is antisymmetric on A , then R_1^{-1} is also antisymmetric.

Proof 1. $\langle x, y \rangle \in A$ is chosen arbitrarily,

$$\langle x, y \rangle \in R_1^{-1} \wedge \langle y, x \rangle \in R_1^{-1} \Rightarrow \langle y, x \rangle \in R_1 \wedge \langle x, y \rangle \in R_1 \Rightarrow x = y$$

Proof 2. Because $R_1 \cap R_1^{-1} \subseteq I_A$, so $R_1^{-1} \cap (R_1^{-1})^{-1} = R_1 \cap R_1^{-1} \subseteq I_A$, it is thus clear that proposition d) is true.

e) If R_1 is transitive on A , then R_1^{-1} is also transitive.

Proof 1. For all $\langle x, y \rangle \in A$, $\langle y, z \rangle \in A$,

$$\langle x, y \rangle \in R_1^{-1} \wedge \langle y, z \rangle \in R_1^{-1} \Rightarrow \langle y, x \rangle \in R_1 \wedge \langle z, y \rangle \in R_1$$

$$\Rightarrow \langle z, x \rangle \in R_1 \Rightarrow \langle x, z \rangle \in R_1^{-1}$$

Proof 2. Because $R_1 \circ R_1 \subseteq R_1$, so $R_1^{-1} \circ R_1^{-1} = (R_1 \circ R_1)^{-1} \subseteq R_1^{-1}$, it is thus evident that proposition e) is true.

3.2 Judgment of the properties of union operation

a) If R_1 and R_2 are all reflexive on A , then $R_1 \cup R_2$ is also reflexive.

Proof 1. $x \in A$ is chosen arbitrarily,

$$\begin{aligned} x \in R_1 \cup R_2 &\Rightarrow x \in R_1 \vee x \in R_2 \Rightarrow \langle x, x \rangle \in R_1 \vee \langle x, x \rangle \in R_2 \\ &\Rightarrow \langle x, x \rangle \in R_1 \cup R_2 \end{aligned}$$

Proof 2. Because R_1 is reflexive, so $I_A \subseteq R_1$, similarly, $I_A \subseteq R_2$, we have $I_A \subseteq R_1 \cup R_2$, therefore, proposition a) is true.

Note that the conditions can be relaxed here, and when R_1 and R_2 at least one of them is reflexive, the conclusion is also established

b) If R_1 and R_2 are all irreflexive on A , then $R_1 \cup R_2$ is also irreflexive.

Proof 1. Assume it is reflexive, for any one $x \in A$,

$$\langle x, x \rangle \in R_1 \cup R_2 \Rightarrow \langle x, x \rangle \in R_1 \vee \langle x, x \rangle \in R_2$$

Obviously, this reasoning contradicts the prerequisite that R_1 and R_2 are all irreflexive, so $R_1 \cup R_2$ is also irreflexive.

Proof 2. R_1, R_2 are all irreflexive on A , we have $I_A \cap R_1 = \emptyset$, $I_A \cap R_2 = \emptyset$, and then $I_A \cap (R_1 \cup R_2) = (I_A \cap R_1) \cup (I_A \cap R_2) = \emptyset$, it is thus evident that proposition b) is true.

c) If R_1 and R_2 are all symmetric on A , then $R_1 \cup R_2$ is also symmetric.

Proof 1. For every $\langle x, y \rangle \in A$,

$$\begin{aligned} \langle x, y \rangle \in R_1 \cup R_2 &\Rightarrow \langle x, y \rangle \in R_1 \vee \langle x, y \rangle \in R_2 \Rightarrow \\ &\langle y, x \rangle \in R_1 \vee \langle y, x \rangle \in R_2 \Rightarrow \langle y, x \rangle \in R_1 \cup R_2 \end{aligned}$$

Proof 2. We have $R_1 = R_1^{-1}$, $R_2 = R_2^{-1}$ according to the prerequisite, and then $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1} = R_1 \cup R_2$.

d) If R_1 and R_2 are all antisymmetric on A , in which case, there are three possibilities for $R_1 \cup R_2$: a) is symmetric; b) is antisymmetric; c) Neither symmetric nor antisymmetric^[14], specific counter examples can refer to the literature [14].

e) If R_1 and R_2 are all transitive on A , then $R_1 \cup R_2$ may be transitive, or it may not be transitive, Counterexamples are as follows:

Example 1: Let $A = \{1, 2\}$, $R_1 = \{\langle 1, 2 \rangle\}$, $R_2 = \{\langle 2, 1 \rangle\}$, it shows R_1 and R_2 are all transitive, but $R_1 \cup R_2$ is not transitive.

Example 2: When $R_1 = R_2$, and then

$$(R_1 \cup R_2) \circ (R_1 \cup R_2) \subseteq R_1 \cup R_2$$

3.3 Judgment of the properties of intersection operation

a) If R_1 and R_2 are all reflexive on A , then $R_1 \cap R_2$ is also reflexive.

Here, we could imitate the proof of above section, omitted.

b) If R_1 and R_2 are all irreflexive on A , then $R_1 \cap R_2$ is also irreflexive.

This proof is similar to above section, omitted here.

c) If R_1 and R_2 are all symmetric on A , then $R_1 \cap R_2$ is also symmetric.

This proof is similar to above section, omitted here.

d) If R_1 and R_2 are all antisymmetric on A , then $R_1 \cap R_2$ is also antisymmetric.

Proof 1. $\langle x, y \rangle \in A$ is chosen arbitrarily,

$$\begin{aligned} \langle x, y \rangle \in R_1 \cap R_2 \wedge \langle y, x \rangle \in R_1 \cap R_2 &\Rightarrow \langle x, y \rangle \in R_1 \wedge \langle y, x \rangle \in R_2 \\ \wedge \langle y, x \rangle \in R_1 \wedge \langle x, y \rangle \in R_2 &\Rightarrow (\langle x, y \rangle \in R_1 \wedge \langle y, x \rangle \in R_1) \wedge \\ (\langle x, y \rangle \in R_2 \wedge \langle y, x \rangle \in R_2) &\Rightarrow x = y \end{aligned}$$

Proof 2. We have $R_1 \cap R_1^{-1} \subseteq I_A$, $R_2 \cap R_2^{-1} \subseteq I_A$ by prerequisite, and then $R_1 \cap R_2 \cap (R_1 \cap R_2)^{-1} \Rightarrow R_1 \cap R_1^{-1} \cap (R_2 \cap R_2^{-1}) \subseteq I_A$.

e) If R_1 and R_2 are all transitive on A , then $R_1 \cap R_2$ is also transitive.

Proof 1. $\langle x, y \rangle \in A, \langle y, z \rangle \in A$ is chosen arbitrarily,

$$\begin{aligned} & \langle x, y \rangle \in R_1 \cap R_2 \wedge \langle y, z \rangle \in R_1 \cap R_2 \\ \Rightarrow & \langle x, y \rangle \in R_1 \wedge \langle x, y \rangle \in R_2 \wedge \langle y, z \rangle \in R_1 \wedge \langle y, z \rangle \in R_2 \\ \Rightarrow & \langle x, y \rangle \in R_1 \wedge \langle y, z \rangle \in R_1 \wedge \langle x, y \rangle \in R_2 \wedge \langle y, z \rangle \in R_2 \\ \Rightarrow & \langle x, z \rangle \in R_1 \wedge \langle x, z \rangle \in R_2 \Rightarrow \langle x, z \rangle \in R_1 \cap R_2 \end{aligned}$$

Proof 2. We have $R_1 \circ R_1 \subseteq R_1, R_2 \circ R_2 \subseteq R_2$, and then

$$\begin{aligned} (R_1 \cap R_2) \circ (R_1 \cap R_2) & \subseteq R_1 \circ R_1 \cap R_2 \circ R_2 \subseteq R_1 \cap R_2 \circ R_1 \cap R_2 \circ R_2 \\ & \subseteq (R_1 \cap R_2) \cap R_1 \circ R_2 \cap R_2 \circ R_1 \subseteq R_1 \cap R_2 \end{aligned}$$

3.4 Judgment of the properties of difference operation

a) If R_1 and R_2 are all reflexive on A , then $R_1 - R_2$ is not reflexive, but it must be irreflexive.

Proof. Because $I_A \subseteq R_2$, then $I_A \cap (R_1 - R_2) = I_A \cap R_1 \cap \bar{R}_2 = \emptyset$

b) If R_1 and R_2 are all irreflexive on A , then $R_1 - R_2$ is also irreflexive.

Proof 1. $x \in A$ is chosen arbitrarily,

$$x \in A \Rightarrow \langle x, x \rangle \notin R_1 \Rightarrow \langle x, x \rangle \notin R_1 - R_2$$

Proof 2. R_1, R_2 are all irreflexive on A , we have $I_A \cap R_1 = \emptyset, I_A \cap R_2 = \emptyset$, and then $I_A \cap (R_1 - R_2) = I_A \cap R_1 \cap \bar{R}_2 = \emptyset$.

c) If R_1 and R_2 are all symmetric on A , then $R_1 - R_2$ is also symmetric.

Proof 1. $\langle x, y \rangle \in A$ is chosen arbitrarily,

$$\begin{aligned} \langle x, y \rangle \in R_1 - R_2 & \Rightarrow \langle x, y \rangle \in R_1 \wedge \langle x, y \rangle \notin R_2 \Rightarrow \\ & \langle y, x \rangle \in R_1 \wedge \langle y, x \rangle \notin R_2 \Rightarrow \langle y, x \rangle \in R_1 - R_2 \end{aligned}$$

Proof 2. We have $R_1 = R_1^{-1}, R_2 = R_2^{-1}$ according to the prerequisite, and then $(R_1 - R_2)^{-1} = R_1^{-1} \cap (\bar{R}_2)^{-1} = R_1 \cap \bar{R}_2$.

Here, $\bar{R}_2 = E_A - R_2$ is the absolute complement, it is closed about symmetry, can be proved.

d) If R_1 and R_2 are all antisymmetric on A , then $R_1 - R_2$ is also antisymmetric.

Proof 1. $\langle x, y \rangle \in A$ is chosen arbitrarily,

$$\begin{aligned} \langle x, y \rangle \in R_1 - R_2 \wedge \langle y, x \rangle \in R_1 - R_2 \\ \Rightarrow \langle x, y \rangle \in R_1 \wedge \langle y, x \rangle \in R_2 \Rightarrow x = y \end{aligned}$$

Proof 2. We have $R_1 \cap R_1^{-1} \subseteq I_A, R_2 \cap R_2^{-1} \subseteq I_A$ and then,

$$(R_1 - R_2) \cap (R_1 - R_2)^{-1} \Rightarrow R_1 \cap \bar{R}_2 \cap R_1^{-1} \cap (\bar{R}_2)^{-1} \subseteq I_A \cap \bar{R}_2 \cap (\bar{R}_2)^{-1} \subseteq I_A$$

e) If R_1 and R_2 are all transitive on A , then $R_1 - R_2$ is not always transitive.

Example 1: Let $A = \{1, 2, 3\}$, $R_1 = \{\langle 2, 2 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle\}$, $R_2 = \{\langle 2, 2 \rangle\}$, we see that R_1 and R_2 are all transitive, and $R_1 - R_2$ is also transitive.

Example 2: Let $A = \{1, 2, 3\}$, $R_1 = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 1, 3 \rangle\}$, $R_2 = \{\langle 1, 3 \rangle\}$, we see that R_1 and R_2 are all transitive, but $R_1 - R_2$ is not transitive.

3.5 Judgment of the properties of composition operation

a) If R_1, R_2 are all reflexive, then $R_1 \circ R_2$ is also reflexive.

Proof 1. For every $x \in A$,

$$x \in A \Rightarrow \langle x, x \rangle \in R_1 \wedge \langle x, x \rangle \in R_2 \Rightarrow \langle x, x \rangle \in R_1 \circ R_2$$

Proof 2. Because R_1 is reflexive, so $I_A \subseteq R_1$, similarly, $I_A \subseteq R_2$,

we have $I_A \circ I_A = I_A \subseteq R_1 \circ R_2$.

b) If R_1 and R_2 are all irreflexive on A , at this moment, there are three possibilities for $R_1 \circ R_2$: a) is reflexive; b) is irreflexive; c) Neither reflexive nor irreflexive^[14], specific counter examples can refer to the literature [14].

c) If R_1 and R_2 are all symmetric on A , at this moment, there are four possibilities for $R_1 \circ R_2$: a) is symmetric; b) is antisymmetric; c) Neither symmetric nor antisymmetric, d) both symmetric and antisymmetric^[14], specific counter examples can refer to the literature [14].

d) If R_1 and R_2 are all antisymmetric on A , the situations of $R_1 \circ R_2$ is same as c).

e) If R_1 and R_2 are all transitive on A , then $R_1 \circ R_2$ is not always transitive, specific counter examples can refer to the literature [14].

3.6 Judgment of the properties of closure operation

a) If R_1 is reflexive, we know that it is reflexive closure, $r(R_1)$ must be reflexive. Also because $I_A \subseteq R_1 \subseteq s(R_1)$, $I_A \subseteq R_1 \subseteq t(R_1)$, so $s(R_1)$, $t(R_1)$ are all reflexive.

b) If R_1 is irreflexive, from the definition of reflexive closure, obviously, $r(R_1)$ is certainly not irreflexive. From R_1 is irreflexive, we have $I_A \cap R_1 = \emptyset$, we can also see $(I_A \cap R_1)^{-1} = \emptyset$, and then $I_A \cap R_1^{-1} = \emptyset$, hence there is $I_A \cap s(R_1) = I_A \cap (R_1 \cup R_1^{-1}) = \emptyset$, then show that $s(R_1)$ is irreflexive. $t(R_1)$ is not necessarily irreflexive, there are three kinds of situations: (1) reflexive; (2) irreflexive; (3) neither reflexive nor irreflexive. Give counterexamples: (1) Let $A = \{1, 2\}$, $R_1 = \{<1, 2>, <2, 1>\}$, we can see R_1 is irreflexive, but $t(R_1) = \{<1, 2>, <2, 1>, <1, 1>, <2, 2>\}$ is reflexive; (2) Let $A = \{1, 2, 3\}$, $R_1 = \{<1, 2>, <2, 3>\}$ is irreflexive, $t(R_1) = \{<1, 2>, <2, 3>, <1, 3>\}$ is also irreflexive; (3) Let $A = \{1, 2, 3\}$, $R_1 = \{<1, 2>, <2, 1>\}$ is irreflexive, but $t(R_1) = \{<1, 2>, <2, 1>, <1, 1>, <2, 2>\}$ is neither reflexive nor irreflexive.

c) If R_1 is symmetric, obviously, its $s(R_1)$ is certainly symmetric. $(r(R_1))^{-1} = (I_A \cup R_1)^{-1} = I_A^{-1} \cup R_1^{-1} = I_A \cup R_1 = r(R_1)$, we know that $r(R_1)$ is symmetric, $t(R_1)$ is also symmetric^[12], detailed proof is in reference [12].

d) If R_1 is antisymmetric, we have $R_1 \cap R_1^{-1} \subseteq I_A$, hence,

$$\begin{aligned} r(R_1) \cap (r(R_1))^{-1} &= (R_1 \cup I_A) \cap (R_1 \cup I_A)^{-1} = (R_1 \cup I_A) \cap (R_1^{-1} \cup I_A) \\ &= (R_1 \cap R_1^{-1}) \cup I_A \subseteq I_A \end{aligned}$$

So $r(R_1)$ is antisymmetric. $s(R_1)$ is certainly symmetric, otherwise it is not called symmetric closure, but when $R_1 = I_A$, is also antisymmetric. $t(R_1)$ is not certainly symmetric, there are three conditions: 1) is symmetric; 2) is antisymmetric; 3) both symmetric and antisymmetric. For example, let $R_1 = I_A$, it is clear that $t(R_1)$ is antisymmetric, $t(R_1) = t(I_A) = I_A$, of course, $t(R_1)$ is symmetric as well as antisymmetric.

e) If R_1 is transitive, we have,

$$\begin{aligned} r(R_1) \circ r(R_1) &= (I_A \cup R_1) \circ (I_A \cup R_1) = I_A \cup R_1 \cup (R_1 \circ R_1) \\ &\subseteq I_A \cup R_1 = r(R_1) \end{aligned}$$

So $r(R_1)$ is transitive. According to the definition of transitive closure, of course, $t(R_1)$ is transitive. $s(R_1)$ is not always transitive^[1], specific counter examples can refer to the literature [1].

5 Conclusion

From above helpful discussion, we could gain much more insights and deepen our knowledge about binary relations, given the solution to some issues relating to properties and operations which is rarely referred or studied. This paper is not very abstruse, but it is a fundamental research and meaningful work, which will be useful for us to learn much deeper theories or further study.

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