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Deducing the form factors for shear used in the calculus of the displacements based on strain energy methods. Mathematical approach for currently used shapes

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Abstract. The paper presents an initial study concerning the form factors for shear, for a rectangular and for a circular cross section, being used an analytical method and a numerical study. The numerical study considers a division of the cross section in small areas and uses the power of the definitions in order to compute the according integrals. The accurate values of the form factors are increasing the accuracy of the displacements computed by the use of the strain energy methods. The knowledge resulted from this study will be used for several directions of development: calculus of the form factors for a ring-type cross section of variable ratio of the inner and outer diameters, calculus of the geometrical characteristics of an inclined circular segment and, using a Bool algebra that operates with geometrical shapes, for an inclined circular ring segment. These shapes may be used to analytically define the geometrical model of a complex composite section, i.e. a ship hull cross section. The according calculus relations are also useful for the development of customized design commands in CAD commercial applications. The paper is a result of the long run development of original computer based instruments in engineering of the authors.

1. Introduction

Several types of models may be developed when a phenomenon is being studied. Analytical models are using the classic knowledge in the field which is based on simplifying hypotheses. These assumptions were useful in order to minimise the amount of calculi by accepting a certain inaccuracy. Nowadays calculus instrument is the computer and it offers speed and accuracy when a certain volume of calculi must be performed. Moreover, it may be used for the integration of the various studies offering an overview regarding the phenomenon under investigation [1]. Regarding the analytical models, the computer may be used to disregard the classic hypotheses, for instance the small displacements assumption, by the use of the numerical integration [2]. In order to use the opportunities offered by the nowadays information technology development, the classic problems must be reformulated starting from the idea that the computer and the computer programming are the main calculus instruments. This approach includes the mathematical background of the problems which must be considered from a new perspective [3].

2. Analytical context and problem formulation

Calculus of the displacements of a given structure may be done using several approaches, for instance the method of initial parameters or using strain energy methods. In what follows, we consider the strain energy methods. An initial way to express the strain energy takes into account the stresses:

$$U^{(\sigma)} = \int_V \frac{\sigma^2}{2\cdot E}\cdot dV\,,\; U^{(\tau)} = \int_V \frac{\tau^2}{2\cdot G}\cdot dV\,, \tag{1}$$

where E is Young's modulus and G is the shear modulus or the modulus of rigidity.

For simple axial loads the normal stress may be computed using the relation $\sigma^N = \dfrac{N}{A}$, where A is the area of the cross section and the according strain energy is

$$U^{(\sigma^N)} = \int_{L_i}\left(\frac{N_i^2(x)}{2\cdot E_i\cdot A_i}\right)\cdot dx\,. \tag{2}$$

For pure bending with M_Y or M_Z, the according normal stresses are $\sigma^{M_Y} = \dfrac{M_Y}{I_Y}\cdot z$, $\sigma^{M_Z} = -\dfrac{M_Z}{I_Z}\cdot y$, where I_Y and I_Z are the second moments of area. The according strain energy relations are

$$U^{(\sigma^{M_Y})} = \int_{L_i}\left(\frac{M_{Y\,i}^2(x)}{2\cdot E_i\cdot I_{Y\,i}}\right)\cdot dx\,,\; U^{(\sigma^{M_Z})} = \int_{L_i}\left(\frac{M_{Z\,i}^2(x)}{2\cdot E_i\cdot I_{Z\,i}}\right)\cdot dx\,. \tag{3}$$

For the T_Y and T_Z shear forces, the shear stresses may be computed using Juravschi's relations $\tau^{(T_Y)} = \dfrac{T_Y}{I_Z}\cdot\left(\dfrac{S_Z}{b_Z}\right)$, $\tau^{(T_Z)} = \dfrac{T_Z}{I_Y}\cdot\left(\dfrac{S_Y}{b_Y}\right)$, where S_Y, S_Z are first moments of area and b_Y, b_Z are widths of the cross section in the current point where the shear stresses are computed. The according strain energy relations are

$$U^{(T_Y)} = \int_{L_i}\left(\frac{k_{Y\,i}}{2\cdot G_i\cdot A_i}\cdot T_{Y\,i}^2(x)\right)\cdot dx\,,\; U^{(T_Z)} = \int_{L_i}\left(\frac{k_{Z\,i}}{2\cdot G_i\cdot A_i}\cdot T_{Z\,i}^2(x)\right)\cdot dx\,, \tag{4}$$

where

$$k_{Y\,i} = \frac{A_i}{I_{Z\,i}^2}\int_{A_i}\left(\frac{S_Z}{b_Z}\right)^2\cdot dA\,,\; k_{Z\,i} = \frac{A_i}{I_{Y\,i}^2}\int_{A_i}\left(\frac{S_Y}{b_Y}\right)^2\cdot dA \tag{5}$$

are designated form factors for shear.

The M_X twisting moment is generating the shear stress $\tau^{(M_X)} = \dfrac{M_X}{I_P}\cdot r$, where I_P is the polar moment of area and the according strain energy is

$$U^{(\tau^{M_X})} = \int_{L_i}\left(\frac{M_{X\,i}^2}{2\cdot G_i\cdot I_{P\,i}}\right)\cdot dx\,. \tag{6}$$

Using the principle of superposition, the energy produced by all internal forces and moments along an interval of L_i length is

$$U = \int_{L_i}\left(\frac{N_i^2}{2\cdot E_i\cdot A_i}\right)\cdot dx + \int_{L_i}\left(\frac{k_{Y\,i}}{2\cdot G_i\cdot A_i}\cdot T_{Y\,i}^2\right)\cdot dx + \int_{L_i}\left(\frac{k_{Z\,i}}{2\cdot G_i\cdot A_i}\cdot T_{Z\,i}^2\right)\cdot dx +$$

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$$+\int_{L_i}\left(\frac{M_{X\,i}^2}{2\cdot G_i\cdot I_{P\,i}}\right)\cdot dx+\int_{L_i}\left(\frac{M_{Y\,i}^2}{2\cdot E_i\cdot I_{Y\,i}}\right)\cdot dx+\int_{L_i}\left(\frac{M_{Z\,i}^2}{2\cdot E_i\cdot I_{Z\,i}}\right)\cdot dx\,. \tag{7}$$

For a structure consisting of 'N' intervals for which the geometrical characteristics and the material constants are identical for each interval while the internal forces and moments may have a certain variation along the interval, the strain energy is [4]:

$$\begin{aligned}U=&\sum_{i=1}^{N}\left[\frac{1}{2\cdot E_i\cdot A_i}\cdot\left(\int_{L_i}N_i^2(x)\cdot dx\right)\right]+\\&+\sum_{i=1}^{N}\left[\frac{k_{Y\,i}}{2\cdot G_i\cdot A_i}\cdot\left(\int_{L_i}T_{Y\,i}^2(x)\cdot dx\right)\right]+\sum_{i=1}^{N}\left[\frac{k_{Z\,i}}{2\cdot G_i\cdot A_i}\cdot\left(\int_{L_i}T_{Z\,i}^2(x)\cdot dx\right)\right]+\\&+\sum_{i=1}^{N}\left[\frac{1}{2\cdot G_i\cdot I_{P\,i}}\cdot\left(\int_{L_i}M_{X\,i}^2(x)\cdot dx\right)\right]+\\&+\sum_{i=1}^{N}\left[\frac{1}{2\cdot E_i\cdot I_{Y\,i}}\cdot\left(\int_{L_i}M_{Y\,i}^2(x)\cdot dx\right)\right]+\sum_{i=1}^{N}\left[\frac{1}{2\cdot E_i\cdot I_{Z\,i}}\cdot\left(\int_{L_i}M_{Z\,i}^2(x)\cdot dx\right)\right].\end{aligned} \tag{8}$$

If we place a unit dummy load along the δ_K unknown displacement, and we remove the real loads, it results the functions of the internal forces and moments $n(x)$, $t_Y(x)$, $t_Z(x)$, $m_X(x)$, $m_Y(x)$ and $m_Z(x)$ on the same interval, 'i', as the real internal forces and moments.

By applying the Reciprocal Work Theorem or Betti's Theorem, it results the expression of the unknown displacement

$$\begin{aligned}\delta_K=&\sum_{i=1}^{N}\left[\frac{1}{(E\cdot A)_i}\cdot\int_{L_i}\left(N_i(x)\cdot n_i(x)\right)\cdot dx\right]+\sum_{i=1}^{N}\left[\left(\frac{k_Y}{G\cdot A}\right)_i\cdot\int_{L_i}\left(T_{Y\,i}(x)\cdot t_{Y\,i}(x)\right)\cdot dx\right]+\\&+\sum_{i=1}^{N}\left[\left(\frac{k_Z}{G\cdot A}\right)_i\cdot\int_{L_i}\left(T_{Z\,i}(x)\cdot t_{Z\,i}(x)\right)\cdot dx\right]+\sum_{i=1}^{N}\left[\frac{1}{(G\cdot I_P)_i}\int_{L_i}\left(M_{X\,i}(x)\cdot m_{X\,i}(x)\right)\cdot dx\right]+\\&+\sum_{i=1}^{N}\left[\frac{1}{(E\cdot I_Y)_i}\int_{L_i}\left(M_{Y\,i}(x)\cdot m_{Y\,i}(x)\right)\cdot dx\right]+\sum_{i=1}^{N}\left[\frac{1}{(E\cdot I_Z)_i}\int_{L_i}\left(M_{Z\,i}(x)\cdot m_{Z\,i}(x)\right)\cdot dx\right].\end{aligned} \tag{9}$$

In most of the 'classic' approaches the students are disregarding the effects of the internal forces. However, using the actual computing instruments and considering the accuracy of the results an important goal of the computer based analytical models, we must take into account the terms where the shear force appears in the previous relation, i.e. the k_Y and k_Z form factors for shear.

3. Discussion

The form factors may be computed analytically for simple shape cross sections. So far we haven't found proofs of the form factors for shear, the according values for simple shapes being taken from one book to another [5].

For complex composite shapes there may be conceived numerical approaches, in this case being necessary a new formulation of the problem.

3.1. Analytical approach for a rectangular section

Let us consider the calculus scheme presented in the following figure. As it may be noticed, we use the symmetry of the domain with respect to the vertical axis. Let us consider a point located at distance z in respect to the horizontal axis. Between the current point and the bottom most boundary of the section is the subdomain for which we consider the first moment of area.

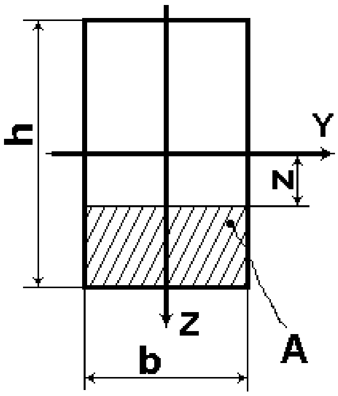

Figure 1. Calculus scheme for a rectangular section.

The first moment of area is:

$$S_Y(z)=\underbrace{\left[b\cdot\left(\frac{h}{2}-z\right)\right]}_{Area}\cdot\underbrace{\left[z+\frac{\left(\frac{h}{2}-z\right)}{2}\right]}_{\substack{Distance\ from\ the\ centroid\\ of\ the\ hatched\ area\\ to\ the\ Y\ centroid\ axis}}=b\cdot\left(\frac{h}{2}-z\right)\cdot\frac{1}{2}\cdot\left(2\cdot z+\frac{h}{2}-z\right)=\frac{b}{2}\cdot\left(\frac{h}{2}-z\right)\cdot\left(\frac{h}{2}+z\right).$$

It results the law of variation:

$$S_Y(z)=\frac{b}{2}\cdot\left(\frac{h^2}{4}-z^2\right)=\frac{b\cdot h^2}{8}\cdot\left(1-4\cdot\frac{z^2}{h^2}\right). \tag{10}$$

The k_Z form factor for shear presented in (5) becomes

$$k_Z=\frac{A}{I_Y^2}\cdot\int_A\left(\frac{S_Y}{b_Y}\right)^2\cdot dA=\frac{b\cdot h}{\left(\frac{b\cdot h^3}{12}\right)^2}\cdot\int_A\left[\frac{\frac{b\cdot h^2}{8}\cdot\left(1-4\cdot\frac{z^2}{h^2}\right)}{b}\right]^2 dA,$$

$$k_Z=\frac{144}{b\cdot h^5}\cdot\frac{h^4}{64}\cdot\int_A\left(1-4\cdot\frac{z^2}{h^2}\right)^2 dA=\frac{9}{4}\cdot\frac{1}{b\cdot h}\cdot\int_A\left(1-8\cdot\frac{z^2}{h^2}+16\cdot\frac{z^4}{h^4}\right)dA,$$

$$k_Z=\frac{9}{4}\cdot\frac{1}{b\cdot h}\cdot\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(1-8\cdot\frac{z^2}{h^2}+16\cdot\frac{z^4}{h^4}\right)\cdot\underbrace{b\cdot dz}_{dA}=\frac{9}{4}\cdot\frac{1}{h}\cdot\left[z\Big|_{-\frac{h}{2}}^{+\frac{h}{2}}-\frac{8}{3\cdot h^2}\cdot z^3\Big|_{-\frac{h}{2}}^{+\frac{h}{2}}+\frac{16}{5\cdot h^4}\cdot z^5\Big|_{-\frac{h}{2}}^{+\frac{h}{2}}\right],$$

$$k_Z=\frac{9}{4}\cdot\frac{1}{h}\cdot\left\{\left[\frac{h}{2}-\left(-\frac{h}{2}\right)\right]-\frac{8}{3\cdot h^2}\cdot\left[\frac{h^3}{8}-\left(-\frac{h^3}{8}\right)\right]+\frac{16}{5\cdot h^4}\cdot\left[\frac{h^5}{32}-\left(-\frac{h^5}{32}\right)\right]\right\},$$

$$k_Z = \frac{9}{4}\cdot\frac{1}{h}\cdot\left(h - \frac{8}{3\cdot h^2}\cdot\frac{h^3}{4} + \frac{16}{5\cdot h^4}\cdot\frac{h^5}{16}\right) = \frac{9}{4}\cdot\frac{1}{h}\cdot\left(h - \frac{2}{3}h + \frac{1}{5}h\right) = \frac{9}{4}\cdot\frac{1}{h}\cdot\frac{15-10+3}{15}\cdot h = \frac{9}{4}\cdot\frac{8}{15}.$$

It results $k_Z = \frac{6}{5}$ and in a similar way one can prove that $k_Y = \frac{6}{5}$. It results the form coefficients for a rectangular shape:

$$k_Y = \frac{6}{5},\ k_Z = \frac{6}{5}. \tag{11}$$

3.2. Analytical approach for a circular section

Let us consider the circular cross section in the figure below. We consider a horizontal line at distance $z > 0$ and, from this line, an infinite small distance, dz, which defines an infinite small area, dA, of a curvilinear rectangular region.

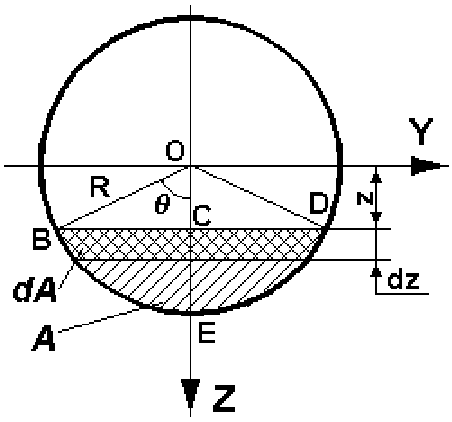

Figure 2. Calculus scheme for a circular section.

The area of the circular sector for a $2\cdot\theta$ central angle is a ratio of the area of the entire circle, $\pi\cdot R^2$, corresponding to a $2\cdot\pi$ central angle. It results

$$A_{OBED} = 2\cdot\theta\cdot\frac{\pi\cdot R^2}{2\cdot\pi} = \theta\cdot R^2. \tag{12}$$

The area of the OBD triangle is $\frac{OC\cdot BD}{2} = \frac{OC\cdot(2\cdot BC)}{2} = OC\cdot BC$.

The OC and BC segments may be expressed with respect to the R radius and the θ angle in the BOC right angled triangle:

$$OC = R\cdot\cos(\theta). \tag{13}$$

$$BC = R\cdot\sin(\theta). \tag{14}$$

This means

$$A_{OBD} = \underbrace{R\cdot\cos(\theta)}_{OC}\cdot\underbrace{R\cdot\sin(\theta)}_{BC} = R^2\cdot\sin(\theta)\cdot\cos(\theta) = \frac{1}{2}\cdot R^2\cdot[2\cdot\sin(\theta)\cdot\cos(\theta)] = \frac{1}{2}\cdot R^2\cdot\sin(2\cdot\theta).$$

The area of the circular segment may be computed by subtracting the area of the OBD triangle from the area of the circular sector, i.e. $A_{BCDE} = A_{OBED} - A_{OBD}$. It results

$$A_{BCDE} = \theta\cdot R^2 - \frac{1}{2}\cdot R^2\cdot\sin(2\cdot\theta) = \frac{R^2}{2}\cdot[2\cdot\theta - \sin(2\cdot\theta)],\ \text{i.e.}$$

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$$A_{BCDE} = \frac{R^2}{2} \cdot [2 \cdot \theta - \sin(2 \cdot \theta)]. \tag{15}$$

The next stage is to evaluate the first moment of area.

The dA infinite small area may be expressed as the area of a curvilinear rectangle, i.e.

$$dA = BD \cdot dz \ . \tag{16}$$

z coordinate may be also expressed with respect to radius R and angle θ in the right angled triangle BOC :

$$z = R \cdot \cos(\theta). \tag{17}$$

The according derivative is:

$$d\,z = -R \cdot \sin(\theta) \cdot d\theta \, . \tag{18}$$

The expression of the infinite small area, dA, becomes

$$dA = |BD \cdot dz| = \underbrace{2 \cdot R \cdot \sin(\theta)}_{BD} \cdot \underbrace{[R \cdot \sin(\theta) \cdot d\theta]}_{dz} = 2 \cdot R^2 \cdot \sin^2(\theta) \cdot d\theta \, . \tag{19}$$

The first moment of area is calculated as an integral for the infinite small area, dA:

$$S_Y(z) = \int_A z \cdot dA = \int_0^\theta \underbrace{R \cdot \cos(\theta)}_{z} \cdot \underbrace{\left[2 \cdot R^2 \cdot \sin^2(\theta) \cdot d\theta\right]}_{dA} = 2 \cdot R^3 \cdot \int_0^\theta \sin^2(\theta) \cdot \cos(\theta) \cdot d\theta \, .$$

$$S_Y(z) = 2 \cdot R^3 \cdot \int_0^\theta \sin^2(\theta) \cdot \cos(\theta) \cdot d\theta = 2 \cdot R^3 \cdot \int_0^\theta \sin^2(\theta) \cdot [\sin(\theta)]' \cdot d\theta = 2 \cdot R^3 \cdot \left[\frac{\sin^3(\theta)}{3}\right]_0^\theta \, .$$

It results the expression:

$$S_Y(z) = \frac{2}{3} \cdot R^3 \cdot \sin^3(\theta). \tag{20}$$

The position of the centroid with respect to the centre of the circle is:

$$Z_G = \frac{S_Y}{A} = \frac{\frac{2}{3} \cdot R^3 \cdot \sin^3(\theta)}{\frac{R^2}{2} \cdot [2 \cdot \theta - \sin(2 \cdot \theta)]} = \frac{4}{3} \cdot R \cdot \frac{\sin^3(\theta)}{2 \cdot \theta - \sin(2 \cdot \theta)} \, . \tag{21}$$

The second moment of area of the circular section is:

$$I_Y = \frac{\pi \cdot d^4}{64} = \frac{\pi \cdot (2 \cdot R)^4}{64} = \frac{16 \cdot \pi}{64} \cdot R^4 = \frac{\pi}{4} \cdot R^4 \, . \tag{22}$$

The width of the section is

$$b_Y = BD = 2 \cdot R \cdot \sin(\theta). \tag{23}$$

In this case, the form factor is

$$k_Z = \frac{A}{I_Y^2} \cdot \int_A \left(\frac{S_Y}{b_Y}\right)^2 \cdot dA = \frac{\pi \cdot R^2}{\left(\frac{\pi}{4} \cdot R^4\right)^2} \cdot \int_A \left[\frac{\frac{2}{3} \cdot R^3 \cdot \sin^3(\theta)}{2 \cdot R \cdot \sin(\theta)}\right]^2 dA \, ,$$

$$k_Z = \frac{16}{\pi} \cdot \frac{1}{R^6} \cdot \int_A \left[\frac{1}{3} \cdot R^2 \cdot \sin^2(\theta)\right]^2 dA = \frac{16}{9 \cdot \pi} \cdot \frac{1}{R^2} \cdot \int_A \sin^4(\theta) \cdot dA.$$

According to (19), $dA = 2 \cdot R^2 \cdot \sin^2(\theta) \cdot d\theta$, it results

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$$k_Z = \frac{16}{9\cdot\pi}\cdot\frac{1}{R^2}\cdot\int_0^{\pi}\sin^4(\theta)\cdot\left[-2\cdot R^2\cdot\sin^2(\theta)\right]\cdot d\theta = -\frac{32}{9\cdot\pi}\cdot\int_0^{\pi}\sin^6(\theta)\cdot d\theta. \tag{24}$$

We consider the following product-to-sum identities:

$$2\cdot\sin(\alpha)\cdot\sin(\varphi) = \cos(\alpha-\varphi)-\cos(\alpha+\varphi) \Rightarrow$$

$$\sin(\alpha)\cdot\sin(\varphi) = \frac{1}{2}\cdot\left[\cos(\alpha-\varphi)-\cos(\alpha+\varphi)\right]. \tag{25}$$

$$2\cdot\cos(\alpha)\cdot\sin(\varphi) = \sin(\alpha+\varphi)-\sin(\alpha-\varphi) \Rightarrow$$

$$\cos(\alpha)\cdot\sin(\varphi) = \frac{1}{2}\cdot\left[\sin(\alpha+\varphi)-\sin(\alpha-\varphi)\right]. \tag{26}$$

We evaluate $\sin^3(\theta)$:

$$\sin^3(\theta) = \sin(\theta)\cdot\sin^2(\theta) = \sin(\theta)\cdot\frac{1-\cos(2\cdot\theta)}{2} = \frac{1}{2}\cdot\left[\sin(\theta)-\cos(2\cdot\theta)\cdot\sin(\theta)\right]. \tag{27}$$

In the (26) identity we replace $\alpha \to 2\cdot\theta$, $\varphi \to \theta$ and it results

$$\cos(2\cdot\theta)\cdot\sin(\theta) = \frac{1}{2}\cdot\left[\sin(2\cdot\theta+\theta)-\sin(2\cdot\theta-\theta)\right] = \frac{1}{2}\cdot\left[\sin(3\cdot\theta)-\sin(\theta)\right]. \tag{28}$$

By replacing (28) in (27), it results

$$\sin^3(\theta) = \frac{1}{2}\cdot\left[\sin(\theta)-\cos(2\cdot\theta)\cdot\sin(\theta)\right] = \frac{1}{2}\cdot\left\{\sin(\theta)-\frac{1}{2}\cdot\left[\sin(3\cdot\theta)-\sin(\theta)\right]\right\} =$$

$$= \left(\frac{1}{2}+\frac{1}{4}\right)\cdot\sin(\theta)-\frac{1}{2}\cdot\frac{1}{2}\cdot\sin(3\cdot\theta) = \frac{3\cdot\sin(\theta)-\sin(3\cdot\theta)}{4}$$

i.e.

$$\sin^3(\theta) = \frac{3\cdot\sin(\theta)-\sin(3\cdot\theta)}{4}. \tag{29}$$

We evaluate $\sin^6(\theta)$:

$$\sin^6(\theta) = \left[\frac{3\cdot\sin(\theta)-\sin(3\cdot\theta)}{4}\right]^2 = \frac{1}{16}\cdot\left[9\cdot\sin^2(\theta)-6\cdot\sin(3\cdot\theta)\cdot\sin(\theta)+\sin^2(3\cdot\theta)\right]. \tag{30}$$

In the (25) identity we replace $\alpha \to 3\cdot\theta$, $\varphi \to \theta$ and it results

$$\sin(3\cdot\theta)\cdot\sin(\theta) = \frac{1}{2}\cdot\left[\cos(3\cdot\theta-\theta)-\cos(3\cdot\theta+\theta)\right] = \frac{1}{2}\cdot\left[\cos(2\cdot\theta)-\cos(4\cdot\theta)\right]. \tag{31}$$

By replacing (31) in (30) it results

$$\sin^6(\theta) = \frac{1}{16}\cdot\left\{9\cdot\frac{1-\cos(2\cdot\theta)}{2}-6\cdot\frac{1}{2}\cdot\left[\cos(2\cdot\theta)-\cos(4\cdot\theta)\right]+\frac{1-\cos(6\cdot\theta)}{2}\right\},$$

$$\sin^6(\theta) = \frac{1}{16}\cdot\left[\frac{9}{2}-\frac{9}{2}\cdot\cos(2\cdot\theta)-3\cdot\cos(2\cdot\theta)+3\cdot\cos(4\cdot\theta)+\frac{1}{2}-\frac{1}{2}\cdot\cos(6\cdot\theta)\right],$$

$$\sin^6(\theta) = \frac{1}{16}\cdot\left[\frac{9+1}{2}-\frac{9+6}{2}\cdot\cos(2\cdot\theta)+3\cdot\cos(4\cdot\theta)-\frac{1}{2}\cdot\cos(6\cdot\theta)\right],$$

$$\sin^6(\theta) = \frac{1}{16}\cdot\left[\frac{10}{2}-\frac{15}{2}\cdot\cos(2\cdot\theta)+3\cdot\cos(4\cdot\theta)-\frac{1}{2}\cdot\cos(6\cdot\theta)\right],$$

$$\sin^6(\theta)=\frac{1}{32}\cdot[10-15\cdot\cos(2\cdot\theta)+6\cdot\cos(4\cdot\theta)-\cos(6\cdot\theta)]. \tag{32}$$

By replacing (32) in the (24) integral, it results

$$k_Z=\frac{32}{9\cdot\pi}\cdot\int_0^{\pi}\sin^6(\theta)\cdot d\theta=\frac{32}{9\cdot\pi}\cdot\int_0^{\pi}\left\{\frac{1}{32}\cdot[10-15\cdot\cos(2\cdot\theta)+6\cdot\cos(4\cdot\theta)-\cos(6\cdot\theta)]\right\}\cdot d\theta,$$

$$k_Z=\frac{10}{9\cdot\pi}\cdot\int_0^{\pi}d\theta-\frac{15}{9\cdot\pi}\cdot\int_0^{\pi}\cos(2\cdot\theta)\cdot d\theta+\frac{6}{9\cdot\pi}\cdot\int_0^{\pi}\cos(4\cdot\theta)\cdot d\theta-\frac{1}{9\cdot\pi}\cdot\int_0^{\pi}\cos(6\cdot\theta)\cdot d\theta,$$

$$k_Z=\frac{10}{9\cdot\pi}\cdot\theta\Big|_0^{\pi}-\frac{5}{3\cdot\pi}\cdot\frac{1}{2}\cdot\sin(2\cdot\theta)\Big|_0^{\pi}+\frac{2}{3\cdot\pi}\cdot\frac{1}{4}\cdot\sin(4\cdot\theta)\Big|_0^{\pi}-\frac{1}{9\cdot\pi}\cdot\frac{1}{6}\cdot\sin(6\cdot\theta)\Big|_0^{\pi},$$

$$k_Z=\frac{10}{9\cdot\pi}\cdot(\pi-0)-\frac{5}{6\cdot\pi}\cdot\overbrace{[\sin(2\cdot\pi)-\sin(0)]}^{0}+\frac{1}{6\cdot\pi}\cdot\overbrace{[\sin(4\cdot\pi)-\sin(0)]}^{0}-$$

$$-\frac{1}{54\cdot\pi}\cdot\underbrace{[\sin(6\cdot\pi)-\sin(0)]}_{0}=\frac{10}{9\cdot\pi}\cdot\pi=\frac{10}{9}.$$

It results $k_Z=\frac{10}{9}$ and in a similar way it can be proved that $k_Y=\frac{10}{9}$. It results the form coefficients for a rectangular shape:

$$k_Y=\frac{10}{9},\ k_Z=\frac{10}{9}. \tag{33}$$

3.3. Numerical approach for a circular section

Let us consider that the area of the circle may be divided in ΔA_i increments according to the calculus scheme presented in the following figure.

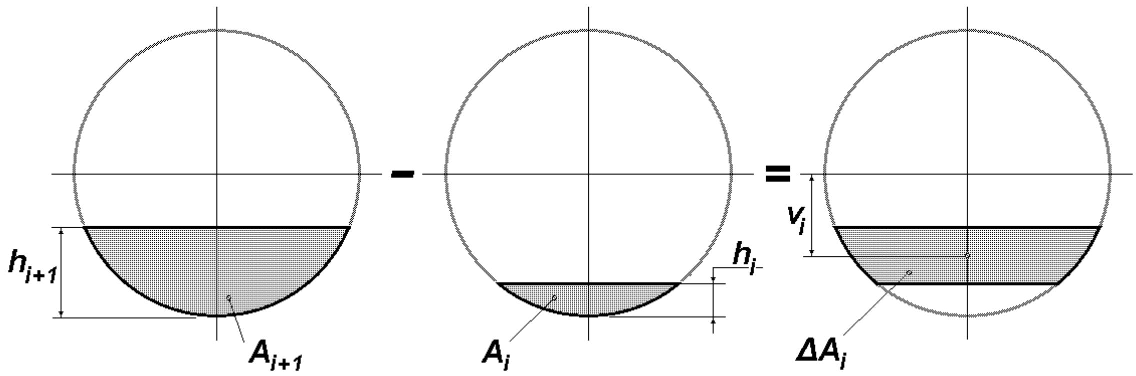

Figure 3. Calculus scheme of the current small area ΔA_i.

The A_{i+1} and A_i areas may be easily computed by applying relation (15) that was deduced for a circular segment. This means that the integrals defined on the area of the circle with respect to dA may be computed as a summation of ΔA_i small areas multiplied by the current function, i.e.

$$I_j=\int_A f_j\cdot dA=\sum_{i=1}^{N} f_{j_i}\cdot\Delta A_i, \tag{34}$$

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where N is the number of ΔA_i small areas in which the area of the circle is divided.

In order to test the procedure which uses small area and its accuracy, we consider some of the geometrical characteristics of a circular section for which we have direct calculus relations, i.e.

$$j=1 \;\Rightarrow\; f_1 = 1 \;\Rightarrow\; A = I_1, \tag{35}$$

$$j=2 \;\Rightarrow\; f_2 = v_i \text{, according to the previous figure } \Rightarrow\; S_Y = I_2, \tag{36}$$

$$j=3 \;\Rightarrow\; f_3 = v_i^2 \text{, according to the previous figure } \Rightarrow\; I_Y = I_3. \tag{37}$$

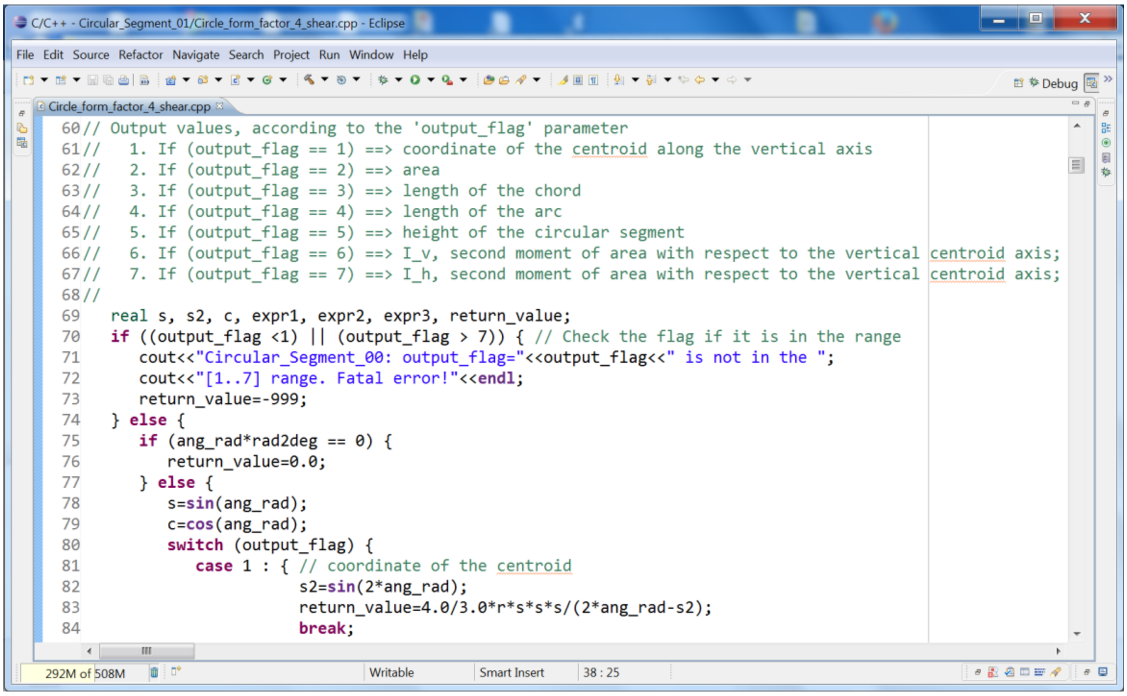
For the k_Z form factor for shear we have

$$k_Z = \frac{A}{I_Y^2} \cdot I_4, \tag{38}$$

where

$$j=4 \;\Rightarrow\; f_4 = \left(\frac{S_{Y_i}}{b_{Y_i}}\right)^2 \;\Rightarrow\; \int_{A_i} \left(\frac{S_Y}{b_Y}\right)^2 \cdot dA = I_4. \tag{39}$$

In order to automatically perform the calculi we have developed a computer code. The first function was developed to compute the attributes of a circular segment.




```
// Output values, according to the 'output_flag' parameter
//   1. If (output_flag == 1) ==> coordinate of the centroid along the vertical axis
//   2. If (output_flag == 2) ==> area
//   3. If (output_flag == 3) ==> length of the chord
//   4. If (output_flag == 4) ==> length of the arc
//   5. If (output_flag == 5) ==> height of the circular segment
//   6. If (output_flag == 6) ==> I_v, second moment of area with respect to the vertical centroid axis;
//   7. If (output_flag == 7) ==> I_h, second moment of area with respect to the vertical centroid axis;
//
   real s, s2, c, expr1, expr2, expr3, return_value;
   if ((output_flag <1) || (output_flag > 7)) { // Check the flag if it is in the range
      cout<<"Circular_Segment_00: output_flag="<<output_flag<<" is not in the ";
      cout<<"[1..7] range. Fatal error!"<<endl;
      return_value=-999;
   } else {
      if (ang_rad*rad2deg == 0) {
         return_value=0.0;
      } else {
         s=sin(ang_rad);
         c=cos(ang_rad);
         switch (output_flag) {
            case 1 : { // coordinate of the centroid
                        s2=sin(2*ang_rad);
                        return_value=4.0/3.0*r*s*s*s/(2*ang_rad-s2);
                        break;
```

Figure 4. Sample computer code of the function which computes the attributes of a circular segment.

According to the previous figure, the output values of the 'Circular_Segment_00' function are identified by the use of the 'output_flag' input parameter, and they are: coordinate of the centroid along the vertical axis, area, length of the chord, length of the arc, 'I_v' - second moment of area with respect to the vertical centroid axis and 'I_h' - second moment of area with respect to the vertical centroid axis.

The second important function of the application is 'Circle_integral_over_area' and, according to the 'output_flag' parameter, it computes the area, the first moment of area with respect to the horizontal axis or the second moment of area with respect to the horizontal axis for a circular cross section, using the (35), (36) and (37) relations, figure 5.

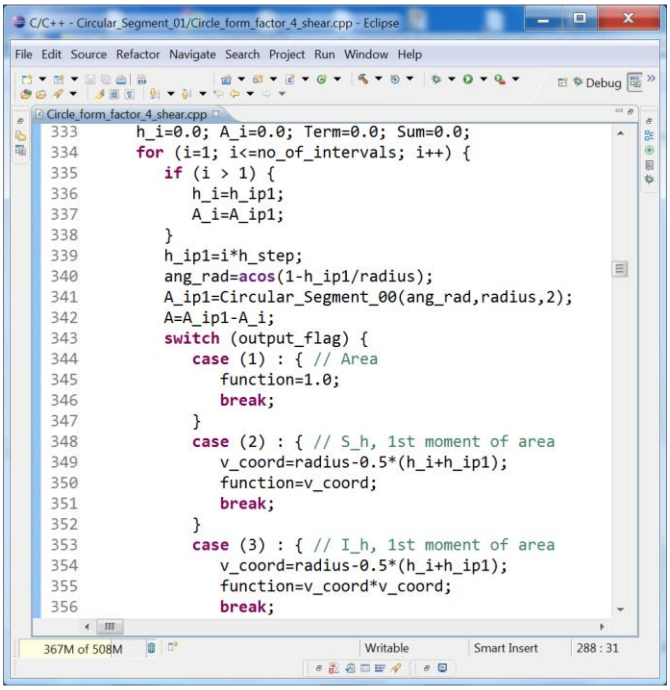


Figure 5. Sample computer code of the function which computes the area, the first moment of area and the second moment of area, of a circular section using the division of the domain in ΔA_i small areas.

The results of the calculus based on the (35), (36) and (37) relations are presented in table 1. As it can be noticed, the errors are very small. Regarding the area, the error is zero because the ΔA_i small areas are computed using the exact calculus relation and the area of the cross section is a summation of these values. Regarding the first moment of area, for the entire section its value is zero. If this value is used to compute the relative error, the denominator would be zero, therefore we consider the absolute error. As it can be noticed, the errors are very small, i.e. $1.36396 \cdot 10^{-10}$. Moreover, the computer code offers the variation of the S_Y first moment of area along the vertical axis, in this way being possible to automatically compute the shear stress using Juravschi's relation, $\tau^{(T_Z)} = \frac{T_Z}{I_Y} \cdot \left(\frac{S_Y}{b_Y} \right)$. Regarding the second moment of area, the relative error is also very small, i.e. for $N = 10$ the according relative error being $\varepsilon = -0.862\ \%$, table 1. These results are accurate, therefore this calculus method based on ΔA_i small areas may be also used for the calculus of I_4, relation (39), and furthermore, k_Y using relation (38).

The calculus of the k_Y form factor for shear is performed in the 'Circle_Form_factor_for_shear' function, a sample code being presented in the following figure.

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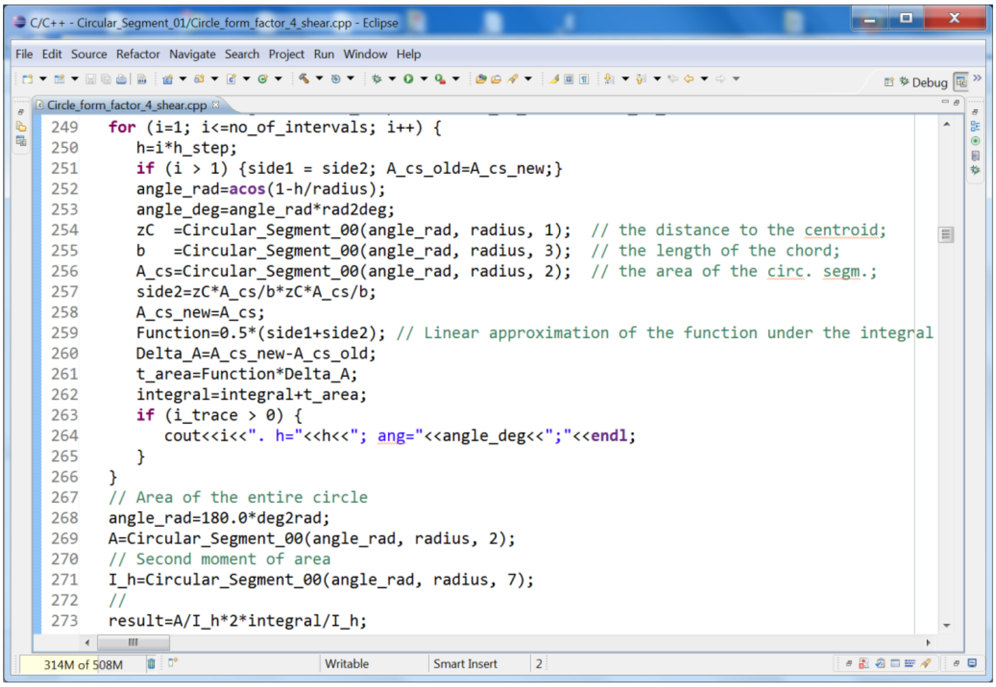

Figure 6. Sample computer code of the function which computes the k_Z form factor for shear.

The results of the calculus are presented in the following table. As it can be noticed, for $N = 10$ the according relative error of the form factor for shear is $\varepsilon = 0.268\ \%$, which is a small value.

Table 1. Results for $R = 100\ mm$

No of intervals, N		10	50	100	1000	Exact
A [mm^2]	Value	31415.9	31415.9	31415.9	31415.9	31415.9
	ε [%]	0 %	0 %	0 %	0 %	
S_Y [mm^3]	Value	$-2.91038\cdot10^{-11}$	$-8.00355\cdot10^{-11}$	$-1.05501\cdot10^{-10}$	$1.36396\cdot10^{-10}$	0
	Absolute ε	$-2.91038\cdot10^{-11}$	$-8.00355\cdot10^{-11}$	$-1.05501\cdot10^{-10}$	$1.36396\cdot10^{-10}$	
I_Y [mm^4]	Value	$7.92171\cdot10^{+7}$	$7.85755\cdot10^{+7}$	$7.85492\cdot10^{+7}$	$7.85399\cdot10^{+7}$	$7.85398\cdot10^{+7}$
	ε [%]	-0.862 %	-0.045 %	-0.011 %	-0.0001 %	
k_Z [-]	Value	$\frac{9.973}{9}$	$\frac{9.998}{9}$	$\frac{9.9997}{9}$	$\frac{10}{9}$	$\frac{10}{9}$
	ε [%]	0.26788 %	0.01067 %	$2.6676\cdot10^{-3}$ %	$2.6667\cdot10^{-5}$ %	

Analysing the results in the previous table it can be noticed that the method based on the ΔA_i small areas yields accurate results. However, the N number of divisions of the circular cross section must be carefully chosen if increased accuracy must be reached.

4. Conclusions

The analytical approaches and the numerical studies presented in the paper offer the same results. The numerical approach is based on the power of the definitions and on the speed and accuracy of the computer based studies.

The knowledge acquired from this study will be used for several purposes. A first direction is to compute the k_Y form factors for a ring-like sections because this cross section is widely used. The form factors must be computed for several $k = \frac{d_{inner}}{D_{outer}}$ ratios, in this way the structural analysts being allowed to select the appropriate value. We plan to use both the analytical and the numerical methods in order to accomplish this goal. Other direction is to create parameterized 'simple shapes' to be used in the calculus of the geometrical characteristics and of the stresses in complex composite sections, i.e. ship hull cross sections. In this way we plan to create algorithms for an inclined circular section and then for an inclined circular ring segment. The according calculus relations, algorithms and computer codes may be also implemented in computer aided design commercial software in order to create customized design commands.

5. Acknowledgement

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