IOP Conf. Series: Materials Science and Engineering **225** (2017) 012018 doi:10.1088/1757-899X/225/1/012018

A detailed comparison study of first order and higher order shear deformation theories in the analysis of laminated composite plate

V M Sreehari[1]
Assistant Professor, School of Mechanical Engineering, SASTRA University, Thanjavur, Tamil Nadu, India-613401.
E-mail: sreehari_vm@mech.sastra.edu

Abstract. The primary aim of the present work is to calculate and compare the response of composite plate using first order and higher order shear deformation theories. The present study initially attempts to develop a finite element formulation for handling the analysis of laminated composite plates. The current study elaborately discusses the formulation that makes an easy programming even for a beginner in this field. Presently, mathematical formulation and Matlab coding using First Order Shear Deformation Theory (FSDT) and Higher Order Shear Deformation Theory (HSDT) had done. Results obtained were compared with the available literature. Parametric study also conducted to clearly understand the variation in results obtained from both FSDT and HSDT.

1. Introduction

Plates form an essential part of many aerospace, marine, and automobile structures. Aircraft and spacecraft structures consist of a large number of flat and curved panel type structural elements. Increased usage of composite laminated plates in crucial structures demands the development of precise theoretical models to predict their response. Plates may be classified into three groups according to the ratio of length/thickness as thick, moderately thick, and thin plates. The behaviour of thin plate structures has been the subject of a number of investigations. The above classification is, of course, conditional because the reference of the plate to one or another group depends on the accuracy of analysis, type of loading, boundary conditions, etc. Thus due to some reason the behaviour of plate response may vary as small deflection or large deflection. The large deflection theory assumes that the deflections are sufficiently large (they can be comparable with the plate thickness or larger), but they should remain small relative to the other dimensions of the plate (except for its thickness). It should be also noted that the deflections of the plate are not assumed to be small, compared with its thickness, but at the same time still sufficiently small to justify an application of the simplified formulas for the plate curvatures. Finally, the large deflection theory deals with finite deflections. However, the relative deformations (strains) are assumed to be small quantities.

Different materials can be combined on a microscopic scale, such as in alloying of metals to form plate like structures, but the resulting material is, for all practical purposes, macroscopically homogeneous, i.e. the components cannot be distinguished by the naked eye and essentially acts together. The word composite in the term of composite material signifies two or more materials are combined on a macroscopic scale to form useful third material. The benefit of composite materials is that, if properly designed, they generally show the best qualities of their components or constituents and frequently certain qualities that neither constituent owns. Also, in many cases, use of composites is more efficient. For example, in aircraft industry, most of the research work is to look for the ways to lower the overall weight of the aircraft without reducing the stiffness and strength of its components. In the past few decades, astonishing advances in sciences and technology have motivated researches to work on new structural materials. The development of composite materials has improved the performance and reliability of structural system. Aerospace structure engineering application requires an accurate prediction of system behavior of structure made up of composites. In the context of optimum design of aircraft components, it is necessary to have a fundamental understanding of their deformation characteristics. In the present work, bending behavior of laminated composite plates will be studied using first and higher order shear deformation theories in detail. The primary aim of the present study is to make a suitable solution technique with finite element method for bending analysis of a laminated composite plate using FSDT and HSDT and find out the variation in results obtained from both FSDT and HSDT. The important goal of the current study is to demonstrate elaborately the formulation that makes an easy programming.

2. Literature review

A few significant works which used FEM are incorporated in this paragraph. These important works using FEM helps for the readers who are learning these FEM concepts and doing formulations using any shear deformation theory. Zienkiewicz [1] studied structural behaviour using FEM. He discussed in detail about von Karman nonlinearity and geometric stiffness matrix associated with the membrane forces. Reddy [2] has described in detail about the laminated composite plates. Analytical and finite element derivations are discussed by Reddy [2] in detail. Solutions for bending, buckling, and vibration are also presented. He presented a good description of the mechanics and associated finite element models of laminated composite structures. Agarwal et al. [3] and Jones [4] presented in detail the fundamental and advance topics related to composite structures. Bhavikatti [5] has discussed the finite element concept and applications to simple structures in detail. Also, application of isoparametric concept to complex problems is discussed. Finite element formulations are made clear by solving simple problems by hand calculation. Sreehari and Maiti [6] presented in detail the introductory concepts, noticeably studied the mathematical formulations of FEM in a buckling and postbuckling problem. Many works are available with descriptions on the computational aspects of FEM. Chandrupatla and Belegundu [7], Ferreria [8], Cook et al. [9], and Kwon and Beng [10] discussed in detail about the finite element coding with numerous examples. Literatures with FEM have employed various shear deformation theories for finding solutions, like classical, first-order, and third-order plate theories.

3. Mathematical formulation

Consider a laminated plate, as in figure 1 as in reference [2], comprising of N orthotropic layers with the principle material co-ordinates (x_1^k, x_2^k, x_3^k) of the k^{th} lamina oriented at an angle θ^k to the laminate co-ordinate, x. The length, width, and thickness of plate are a, b, and h respectively. The co-ordinate system has its origin at the corner of the plate on the mid plane. The z-axis is taken positive downward from the mid plane.

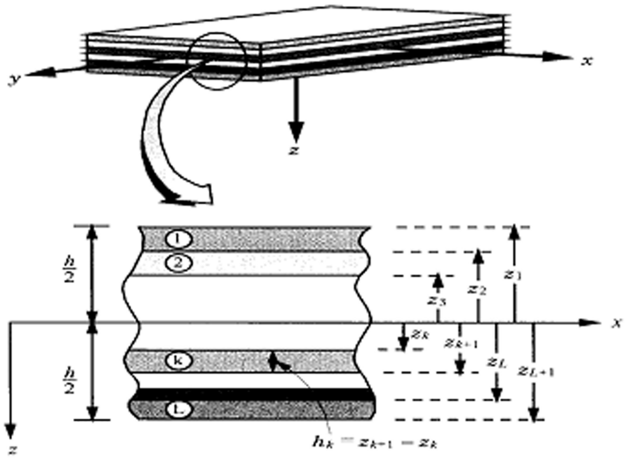

Figure 1. Coordinate system and layer numbering used for a laminated plate.

3.1 Displacement field

A simple higher order shear deformation theory in which transverse shear strains are assumed to be parabolically distributed across the plate thickness is considered initially. The displacement components are assumed to be in the form:

$$\begin{bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} u_o(x,y,z,t) \\ v_o(x,y,z,t) \\ w_o(x,y,z,t) \end{bmatrix} + z\begin{bmatrix} \phi_x(x,y,t) \\ \phi_y(x,y,t) \\ 0 \end{bmatrix} + z^2\begin{bmatrix} \beta_x(x,y,t) \\ \beta_y(x,y,t) \\ 0 \end{bmatrix} + z^3\begin{bmatrix} \psi_x(x,y,t) \\ \psi_y(x,y,t) \\ 0 \end{bmatrix} \tag{1}$$

Where *u, v, and w* are the displacement components in *x, y,* and *z* directions respectively; u_0, v_0, w_0 are the displacements of a point on the mid plane (*x ,y*, 0). ϕ_x, ϕ_y are the rotations of the cross-section perpendicular to *x* and *y* axes respectively. The parameters $\beta_x, \beta_y, \psi_x, \psi_y$ are the higher order terms in Taylor's series expansion and they represent higher order transverse cross sectional modes. For the case of FSDT, displacement field will be as shown below,

$$\begin{bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} u_o(x,y,z,t) \\ v_o(x,y,z,t) \\ w_o(x,y,z,t) \end{bmatrix} + z\begin{bmatrix} \phi_x(x,y,t) \\ \phi_y(x,y,t) \\ 0 \end{bmatrix}$$

3.2 Strain-displacement relations

The linear strain-displacement relations are used in formulating the governing differential equations and are given as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2}$$

3.3 Constitutive relations

For laminate composed of orthotropic layers, with their $x_1 x_2$-plane oriented arbitrarily with *xy*-plane (x_3 = 0), the transverse stresses (σ_{zx}, σ_{yz}) are also zero. An orthotropic material is characterized by nine elastic moduli and has three planes of elastic symmetry. Under the assumption that material behaves linearly elastic, the constitutive relation for each lamina can be written as:

$$\{\sigma\} = [Q]\{\varepsilon\} \tag{3}$$

Where the components of Compliance matrix, Q are expressed in terms of material properties and are given by the equation

$$Q_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}, Q_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})}, Q_{12} = \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})}, \tag{4}$$

$$Q_{44} = G_{23}, Q_{55} = G_{13}, Q_{66} = G_{12}$$

Stress-strain relations in the local co-ordinate system can be expressed as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}_k \tag{5}$$

Where, $\overline{Q}_{ij}$'s are transformed reduced stiffness coefficients and expressed as:

$$\begin{Bmatrix} \overline{Q}_{11} \\ \overline{Q}_{12} \\ \overline{Q}_{22} \\ \overline{Q}_{16} \\ \overline{Q}_{26} \\ \overline{Q}_{66} \\ \overline{Q}_{44} \\ \overline{Q}_{45} \\ \overline{Q}_{55} \end{Bmatrix}_k = \begin{bmatrix} m^4 & 2m^2n^2 & n^4 & 4m^2n^2 & 0 & 0 \\ m^2n^2 & m^4 + n^4 & m^2n^2 & -4m^2n^2 & 0 & 0 \\ n^4 & 2m^2n^2 & m^4 & 4m^2n^2 & 0 & 0 \\ m^3n & mn^3 - m^3n & -m^3n & -2mn(m^2 - n^2) & 0 & 0 \\ mn^3 & m^3n - mn^3 & -mn^3 & 2mn(m^2 - n^2) & 0 & 0 \\ m^2n^2 & -2m^2n^2 & m^2n^2 & (m^2 - n^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m^2 & n^2 \\ 0 & 0 & 0 & 0 & -mn & mn \\ 0 & 0 & 0 & 0 & n^2 & m^2 \end{bmatrix}_k \begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{22} \\ Q_{66} \\ Q_{44} \\ Q_{55} \end{Bmatrix} \tag{6}$$

where m = cosine θ and n =sine θ

3.4 Formulation for finite element method for FSDT

The strain–displacement relations given above are written using FSDT as:

$$\varepsilon_{xx} = \varepsilon_1 = \frac{\partial u}{\partial x} = \varepsilon_1^0 + zk_1^0$$

$$\varepsilon_{yy} = \varepsilon_2 = \frac{\partial v}{\partial y} = \varepsilon_2^0 + zk_2^0$$

$$\gamma_{xy} = \varepsilon_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \varepsilon_6^0 + zk_6^0$$

$$\gamma_{yz} = \varepsilon_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \varepsilon_4^0$$

$$\gamma_{zx} = \varepsilon_5 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \varepsilon_5^0$$

Where
$$\varepsilon_1^0 = \frac{\partial u_0}{\partial x}, \varepsilon_2^0 = \frac{\partial v_0}{\partial y}, \varepsilon_6^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$$
$$\varepsilon_4^0 = \frac{\partial w_0}{\partial y} + \phi_y, \varepsilon_5^0 = \frac{\partial w_0}{\partial x} + \phi_x$$
$$k_1^0 = \frac{\partial \phi_x}{\partial x}, k_2^0 = \frac{\partial \phi_y}{\partial y}, k_6^0 = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$

The linear strain vector given in above equation can also be expressed in terms of midplane strain vector, $\{\overline{\varepsilon}\}$

$$\{\varepsilon\}_{5\times1} = [T]_{5\times8}\{\overline{\varepsilon}\}_{8\times1}$$

Where,

$$\{\overline{\varepsilon}\} = \left\{\varepsilon_1^0 \quad \varepsilon_2^0 \quad \varepsilon_6^0 \quad k_1^0 \quad k_2^0 \quad k_6^0 \quad \varepsilon_4^0 \quad \varepsilon_5^0\right\}^T$$ and

$$[T] = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also,
$$\{\overline{\varepsilon}\}_{8\times1} = [L]_{8\times5}\{\Delta\}_{5\times1}$$
Where,

$$\{\Delta\} = \left\{u_0 \quad v_0 \quad w_0 \quad \phi_x \quad \phi_y\right\}^T$$

and

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 1 \\ 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

$$D = T'\overline{Q}T$$

$$D = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & z\overline{Q}_{11} & z\overline{Q}_{12} & z\overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & z\overline{Q} & z\overline{Q}_{22} & z\overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & z\overline{Q}_{16} & z\overline{Q}_{26} & z\overline{Q}_{16} & 0 & 0 \\ z\overline{Q}_{11} & z\overline{Q}_{12} & z\overline{Q}_{16} & z^2\overline{Q}_{11} & z^2\overline{Q}_{12} & z^2\overline{Q}_{16} & 0 & 0 \\ z\overline{Q}_{12} & z\overline{Q} & z\overline{Q}_{26} & z^2\overline{Q}_{12} & z^2\overline{Q}_{22} & z^2\overline{Q}_{26} & 0 & 0 \\ z\overline{Q}_{16} & z\overline{Q}_{26} & z\overline{Q}_{66} & z^2\overline{Q} & z^2\overline{Q}_{26} & z^2\overline{Q}_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}$$

IOP Conf. Series: Materials Science and Engineering **225** (2017) 012018 doi:10.1088/1757-899X/225/1/012018

3.5 Introducing the HSDT

There are nine dependent unknowns in the displacement field given by equation (1). The number of dependent unknowns can be reduced by imposing the traction-free boundary conditions given by equations below on the top and bottom faces of the laminate.

$$\sigma_{yz}(x, y, \pm h/2) = 0 \text{ and } \sigma_{zx}(x, y, \pm h/2) = 0 \quad \sigma_{yz}$$

If the transverse shear stresses are to vanish at the bounding planes of the plate ($z = \pm h/2$), the transverse shear strains, γ_{yz} and γ_{xz} must also vanish there, i.e.,

$$\gamma_{yz}(x, y, \pm h/2) = 0 \text{ and } \gamma_{zx}(x, y, \pm h/2) = 0$$

Using strain-displacement relations given by equations (2) and displacement field given by equation (1) in above equation, parameters $\beta_x, \beta_y, \psi_x, \psi_y$ can be determined in the form:

$$\psi_x = -\frac{4}{3h^2}\left[\phi_x + \frac{\partial w_0}{\partial x}\right], \psi_y = -\frac{4}{3h^2}\left[\phi_y + \frac{\partial w_0}{\partial y}\right], \beta_x = \beta_y = 0$$

Using this equation, the displacement field given in equation (1) can now be expressed in terms of five dependent unknowns $(u_0, v_0, w_0, \phi_x, \phi_y)$ .The modified displacement field is now written in terms of $(u_0, v_0, w_0, \phi_x, \phi_y)$.

$$\begin{bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{bmatrix} = \begin{bmatrix} u_o(x,y,z,t) \\ v_o(x,y,z,t) \\ w_o(x,y,z,t) \end{bmatrix} + z \begin{bmatrix} \phi_x(x,y,t) \\ \phi_y(x,y,t) \\ 0 \end{bmatrix} - c_1 z^3 \begin{bmatrix} \phi_x + \frac{\partial w_0}{\partial x} \\ \phi_y + \frac{\partial w_0}{\partial y} \\ 0 \end{bmatrix}$$

The significance of constant c_1 is that it facilitates the representation of FSDT and HSDT through same equation. For $c_1 = 4/3h^2$, equation is the case of HSDT which contains the same unknown parameters as in the case of FSDT. For $c_1 = 0$, equation above is for the case of FSDT.

Equation (1) can be now written as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^3 \\ \varepsilon_{yy}^3 \\ \gamma_{xy}^3 \end{Bmatrix}$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{Bmatrix}$$

Introducing $c_1 = 4/3h^2$ and $c_2 = 3c_1$

IOP Conf. Series: Materials Science and Engineering **225** (2017) 012018 doi:10.1088/1757-899X/225/1/012018

Where,

$$\begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial v_0}{\partial x} + \dfrac{\partial u_0}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_y}{\partial x} + \dfrac{\partial \phi_x}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^3 \\ \varepsilon_{yy}^3 \\ \gamma_{xy}^3 \end{Bmatrix} = -c_1 \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_y}{\partial x} + \dfrac{\partial \phi_x}{\partial y} \end{Bmatrix}$$

and

$$\begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} \phi_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{Bmatrix} = -c_1 \begin{Bmatrix} \phi_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix}$$

3.6 Formulation for finite element method for HSDT

The strain–displacement relations are written using HSDT as:

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_1 = \frac{\partial u}{\partial x} = \varepsilon_1^0 + z k_1^0 - c_1 k_1^1 z^3 \\
\varepsilon_{yy} &= \varepsilon_2 = \frac{\partial v}{\partial y} = \varepsilon_2^0 + z k_2^0 - c_1 k_2^1 z^3 \\
\gamma_{xy} &= \varepsilon_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \varepsilon_6^0 + z k_6^0 - c_1 k_6^1 z^3 \\
\gamma_{yz} &= \varepsilon_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \varepsilon_5^0 - 3 c_1 k_5^2 z^2 \\
\gamma_{zx} &= \varepsilon_5 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \varepsilon_4^0 - 3 c_1 k_4^2 z^2
\end{aligned}$$

where,

$$\begin{aligned}
&\varepsilon_1^0 = \frac{\partial u_0}{\partial x}, \varepsilon_2^0 = \frac{\partial v_0}{\partial y}, \varepsilon_6^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\
&\varepsilon_5^0 = \frac{\partial w_0}{\partial y} + \phi_y, \varepsilon_4^0 = \frac{\partial w_0}{\partial x} + \phi_x \\
&k_1^0 = \frac{\partial \phi_x}{\partial x}, k_2^0 = \frac{\partial \phi_y}{\partial y}, k_6^0 = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \\
&k_1^1 = \frac{\partial \phi_x}{\partial x} + \frac{\partial \theta_x}{\partial x}, k_2^1 = \frac{\partial \phi_y}{\partial y} + \frac{\partial \theta_y}{\partial y}, k_6^1 = \frac{\partial \phi_y}{\partial x} + \frac{\partial \theta_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + \frac{\partial \theta_x}{\partial y} \\
&k_4^2 = \phi_x + \theta_x, k_5^2 = \phi_y + \theta_y,
\end{aligned}$$

Similarly as above ,the linear strain vector given in above equation can also be expressed in terms of midplane strain vector, $\{\overline{\varepsilon}\}$

$$\{\varepsilon\}_{5\times 1} = [T]_{5\times 13}\{\overline{\varepsilon}\}_{13\times 1}$$

$$\{\overline{\varepsilon}\} = \left\{ \varepsilon_1^0 \quad \varepsilon_2^0 \quad \varepsilon_6^0 \quad k_1^0 \quad k_2^0 \quad k_6^0 \quad k_1^1 \quad k_2^1 \quad k_6^1 \quad \varepsilon_4^0 \quad \varepsilon_5^0 \quad k_4^2 \quad k_5^2 \right\}^T$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & z^2 \end{bmatrix}$$

$$\{\overline{\varepsilon}\}_{13\times 1} = [L]_{13\times 7}\{\Delta\}_{7\times 1}$$

Where,

$$\{\Delta\} = \left\{u_0 \quad v_0 \quad w_0 \quad \phi_x \quad \phi_y \quad \theta_x \quad \theta_y\right\}^T$$

and,

$$[L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & c_1\frac{\partial}{\partial x} & 0 & c_1\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & c_1\frac{\partial}{\partial y} & 0 & c_1\frac{\partial}{\partial y} \\ 0 & 0 & 0 & c_1\frac{\partial}{\partial y} & c_1\frac{\partial}{\partial x} & c_1\frac{\partial}{\partial y} & c_1\frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3c_1 & 0 & 3c_1 & 0 \\ 0 & 0 & 0 & 0 & 3c_1 & 0 & 3c_1 \end{bmatrix}$$

$$D = T'\overline{Q}T$$

3.7 Potential energy of the laminate

The present analysis involves structural displacement due to external mechanical loading. The total energy of the system can thus be considered as the strain energy due to mechanical loading. Thus, the potential energy of the laminated composite plate undergoing deformation is given as
Potential Energy=Strain Energy

$$U = \frac{1}{2}\int \{\varepsilon\}^T \{\sigma\} dV$$

The stress strain relation can be written as:

$$\{\sigma\} = [\overline{Q}]\{\varepsilon\}$$

Using the equations the equation for potential energy can be written as

$$U = \frac{1}{2}\int \{\varepsilon\}^T \{\overline{Q}\varepsilon\} dV = \frac{1}{2}\int \overline{\varepsilon}^T T^T [\overline{Q} T \overline{\varepsilon}] dV$$

Or

$$U = \frac{1}{2}\int \{\bar{\varepsilon}\}^T [D]\{\bar{\varepsilon}\} dA$$

Where

$$[D] = \sum_{k=1}^{N} \int_{zk-1}^{zk} [T]^T [\bar{Q}][T] dz$$

Thus the expression for potential energy becomes

$$U = \frac{1}{2}\int [\{\Delta\}^T [L]^T [D][L]\{\Delta\}] dA$$

3.8 Solution method

Solution methodologies for present anlaysis are presented. Also the implementation of finite element method with 8- noded isoparametric elements is presented.

The domain is divided into number of sub-domains that are known as finite elements. These elements are connected at various nodes. For the finite element analysis,

$$U = \sum_{e=1}^{NE} U^{(e)}$$

$$U = \sum_{e=1}^{NE} \frac{1}{2}\int [\{\Delta\}^{(e)T} [L]^T [D][L]\{\Delta\}^{(e)}] dA$$

Where NE is the number of elements used for meshing the plate

The displacement vector Δ can be written in terms of shape functions N_i and displacement vector, q for an element as $\{\Delta\}^{(e)} = [N_i]^{(e)}\{q\}^{(e)}$

On substituting, element potential energy can be written as

$$U = \sum_{e=1}^{NE} \frac{1}{2}\int \left(\{q\}^{(e)T} \{N\}^{(e)T} [L]^T [D][L]\{N\}^{(e)} \{q\}^{(e)}\right) dA$$

Element potential energy can be written as

$$U^{(e)} = \frac{1}{2}\int \left(\{q\}^{(e)T} \{B\}^{(e)T} [D]\{B\}^{(e)} \{q\}^{(e)}\right) dA^{(e)}$$

$$[B]^{(e)} = [L][N]^{(e)}$$

Where $[B]^{(e)} = \begin{bmatrix} B_1 & B_2 & B_3 & \ldots\ldots & B_{NN} \end{bmatrix}$

Element bending stiffness matrix is defined as $K^{(e)} = \int B^{(e)T} D B^{(e)} dA^{(e)}$

Thus finally the elemental potential energy can be written as $U^{(e)} = \frac{1}{2} q^{(e)T} K^{(e)} q^{(e)}$

Now, $K^{(e)}$ is computed numerically by transforming the existing coordinate system to natural coordinate system ξ and η ,and then can be written as:

$$K_{ij}^{(e)} = \int_{-1}^{-1}\int_{-1}^{-1} B_i^T D B_j \det J d\xi d\eta$$

Where, J is the Jacobian Matrix and is given by

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

When numerical integration is adopted, the element matrix of equation becomes:

$$K_{ij}^{(e)} = \frac{1}{2}\sum_{p=1}^{N}\sum_{q=1}^{N} W_p W_q B_i^T D B_j \det J$$

Where W_p, W_q are the weights used in the Gaussian quadrature and work done:

$$W = \iint w_p w_q dx dy$$

4. Results and discussions

Finite element method's results for laminated composite plates are obtained by analyzing the formulation explained in previous section and programming in MATLAB. An eight noded C_0, isoparametric element has been employed for discretization of the laminate. For the FSDT, a shear correction factor 5/6 has been used. Based on convergence study, a (12 × 12) mesh has been used in most cases of later study. In all problems considered, the individual layers are taken to be of equal thickness. A variety of problems is studied and is and compared the result to the existing results. The finite element method provides a numerical solution to a complex problem, it may therefore be expected that the solution must converge to the exact solution under certain circumstances. It can be shown that the displacement formulation of the method leads to be upper bound to the actual stiffness of the structure. Hence as the mesh is made finer, the solution should converge to the correct result. Non dimensional results are presented. The non-dimensionality used for transverse deflection is $\overline{w} = w(\frac{100h^3E_2}{q_0a^4})$

Convergence of the solution with refinement in mesh for four layered, symmetric and anti-symmetric cross-ply laminate with *a/h* ratio 10 to 100 is shown in table 1. Similarly convergence of solution with refinement in mesh for four layered, symmetric and anti-symmetric angle-ply laminate with *a/h* ratio 10 to 100 is shown in table 2. As the number of mesh increases the convergence of the results is found to be fairly accurate (also indicated as percentage variation in table 3). It is clear from the obtained results that thick plates have high deflection and deflection becomes almost constant after *a/h* ratio of 40. As the number of layers increases, the deflection becomes almost constant. From the table 1 and 2, it can be concluded that deflections decreases as the *a/h* ratio is increased or number of layers are increased.

IOP Conf. Series: Materials Science and Engineering **225** (2017) 012018 doi:10.1088/1757-899X/225/1/012018

Table 1. Non-dimensional central deflection for symmetric and antisymmetric cross-ply, simply supported-1, subjected to sinusoidal load.

a/h	Mesh size	0/90/90/0	0/90/0/90
10	2x2	0.61078599	0.63179162
	4x4	0.66084344	0.67825433
	6x6	0.66220033	0.67967253
	8x8	0.66240051	0.67988967
	12x12	0.66246599	0.67996510
	16x16	0.66247432	0.67997649
20	2x2	0.46912938	0.52907672
	4x4	0.49103859	0.54983221
	6x6	0.49103351	0.54978842
	8x8	0.49101430	0.54976667
	12x12	0.49100115	0.54975500
	16x16	0.49099687	0.54975188
30	2x2	0.43560225	0.49865843
	4x4	0.45819491	0.52598234
	6x6	0.45800222	0.52573235
	8x8	0.45794732	0.52566923
	12x12	0.45792116	0.52564191
	16x16	0.45791485	0.52563615
40	2x2	0.41897476	0.47785934
	4x4	0.44652561	0.51756527
	6x6	0.44632058	0.51730901
	8x8	0.44625650	0.51723449
	12x12	0.44622675	0.51720223
	16x16	0.44621994	0.51719562
50	2x2	0.40730782	0.45964784
	4x4	0.44105045	0.51359891
	6x6	0.44088891	0.51340659
	8x8	0.44082313	0.51332980
	12x12	0.44079241	0.51329577
	16x16	0.44078554	0.51328882
60	2x2	0.39765708	0.44250455
	4x4	0.43801956	0.51137254
	6x6	0.43792978	0.51128330
	8x8	0.43786524	0.51120817
	12x12	0.43783448	0.51117365
	16x16	0.43782769	0.51116658
70	2x2	0.38909467	0.42602464
	4x4	0.43613903	0.50995674
	6x6	0.43614101	0.50999967
	8x8	0.43607919	0.50992837
	12x12	0.43604881	0.50989400
	16x16	0.43604216	0.50988690
80	2x2	0.38130352	0.41013982
	4x4	0.43486581	0.50896293
	6x6	0.43497686	0.50916328
	8x8	0.43491874	0.50909726
	12x12	0.43488890	0.50906339
	16x16	0.43488242	0.50905632
90	2x2	0.37417221	0.39489880
	4x4	0.43393949	0.50820534
	6x6	0.43417607	0.50858662
	8x8	0.43412239	0.50852700
	12x12	0.43409321	0.50849385
	16x16	0.43408689	0.50848685
100	2x2	0.36765554	0.38038078
	4x4	0.43322246	0.50758606
	6x6	0.43360083	0.50817094
	8x8	0.43355225	0.50811870
	12x12	0.43352378	0.50808641
	16x16	0.43351762	0.50807949

Table 2. Non-dimensional transverse deflection for angle-ply, simply supported-2, sinusoidal load.

a/h	Mesh size	-45/45/45/-45	-45/45/-45/45
10	4x4	0.4970194	0.4546066
	6x6	0.4989648	0.4553509
	8x8	0.4989408	0.4552053
	10x10	0.4988180	0.4550849
	12x12	0.4987020	0.4550050
20	4x4	0.3541954	0.3257407
	6x6	0.3563154	0.3252877
	8x8	0.3564358	0.3249972
	10x10	0.3563869	0.3248496
	12x12	0.3563067	0.3247644
30	4x4	0.3247229	0.3016496
	6x6	0.3282545	0.3011180
	8x8	0.3287686	0.3008477
	10x10	0.3288858	0.3007129
	12x12	0.3288884	0.3006348
40	4x4	0.3127310	0.2930945
	6x6	0.3177112	0.2926053
	8x8	0.3186273	0.2923696
	10x10	0.3189232	0.2922512
	12x12	0.3190184	0.2921813
50	4x4	0.3059923	0.2890616
	6x6	0.3123527	0.2886311
	8x8	0.3136399	0.2884272
	10x10	0.3141066	0.2883240
	12x12	0.3142943	0.2882620
60	4x4	0.3014056	0.2868218
	6x6	0.3090950	0.2864505
	8x8	0.3107194	0.2862729
	10x10	0.3113414	0.2861828
	12x12	0.3116166	0.2861278
70	4x4	0.2978740	0.2854335
	6x6	0.3068662	0.2851214
	8x8	0.3088003	0.2849651
	10x10	0.3095609	0.2848858
	12x12	0.3099158	0.2848370
80	4x4	0.2949249	0.2844999
	6x6	0.3052113	0.2842489
	8x8	0.3074330	0.2841101
	10x10	0.3083169	0.2840397
	12x12	0.3087436	0.2839961
90	4x4	0.2923261	0.2838297
	6x6	0.3039046	0.2836439
	8x8	0.3063993	0.2835195
	10x10	0.3073939	0.2834564
	12x12	0.3078847	0.2834172
100	4x4	0.2899556	0.2833214
	6x6	0.3028248	0.2832059
	8x8	0.3055820	0.2830939
	10x10	0.3066771	0.2830367
	12x12	0.3072254	0.2830013

Table 3. Convergence of FEM solution for different mesh size for symmetric cross-ply laminate denoting the percentage change in last two values.

Lamination Scheme	a/h	Mesh size 2x2	4x4	6x6	8x8	Percentage change in last two values
0/90/90/0	10	0.6107	0.6608	0.6622	0.6624	-0.0320%
	30	0.4356	0.4581	0.4580	0.4579	0.0218%
	100	0.3676	0.4332	0.4336	0.4335	0.0236%
0/90/0	10	0.6781	0.7383	0.7395	0.7396	-0.0134%
	30	0.4398	0.4672	0.4670	0.4669	0.0214%
	100	0.3969	0.4340	0.4344	0.4344	0.0%

The Matlab coding is now being extended to higher order theories (as presented in Table 4) and presently the values of non-dimensional central deflections under the action of transverse loads are got. From the results obtained now in HSDT, it is clear that the third-order theory (TSDT) gives more accurate results for deflections when compared to the first-order shear deformation plate theory with $K = 5/6$. It is known that the shear correction factor K depends on the lamina properties and the stacking sequence. The fact that no shear correction coefficients are needed in the third-order theory makes it more convenient to use. In general, the equilibrium-derived transverse shear stresses compare more favourably with the elasticity solution than those obtained from the constitutive equations for equivalent single-layer theories. There is quite difference between FSDT and HSDT results for thick plates while for thin plates both the theories predict similar behaviour. Effect of transverse shear strain is thus noticed. Figure 2 contains plots of non-dimensionalized centre deflection to thickness ratio a/h for a square, symmetric cross-ply laminate (0/90/90/0) under sinusoidal distributed load. Compared to the elasticity solution, the third-order theory underpredicts deflection by less while the first-order theory underpredicts by higher amounts. When the plate is thick, the difference between FSDT and HSDT values are large. This means that the plate behaviour prediction will be more unsafe by FSDT than by HSDT in such cases.

Table 4. Non-dimensionalised deflection for FSDT and HSDT.

a/h	Source	Non-dimensionalised deflection*100	Reference [2]
10	FSDT	0.6624	0.663
	TSDT	0.7141	0.715
20	FSDT	0.4909	0.4912
	TSDT	0.5055	0.506
100	FSDT	0.4335	0.4337
	TSDT	0.4343	0.434
	CLPT		0.431

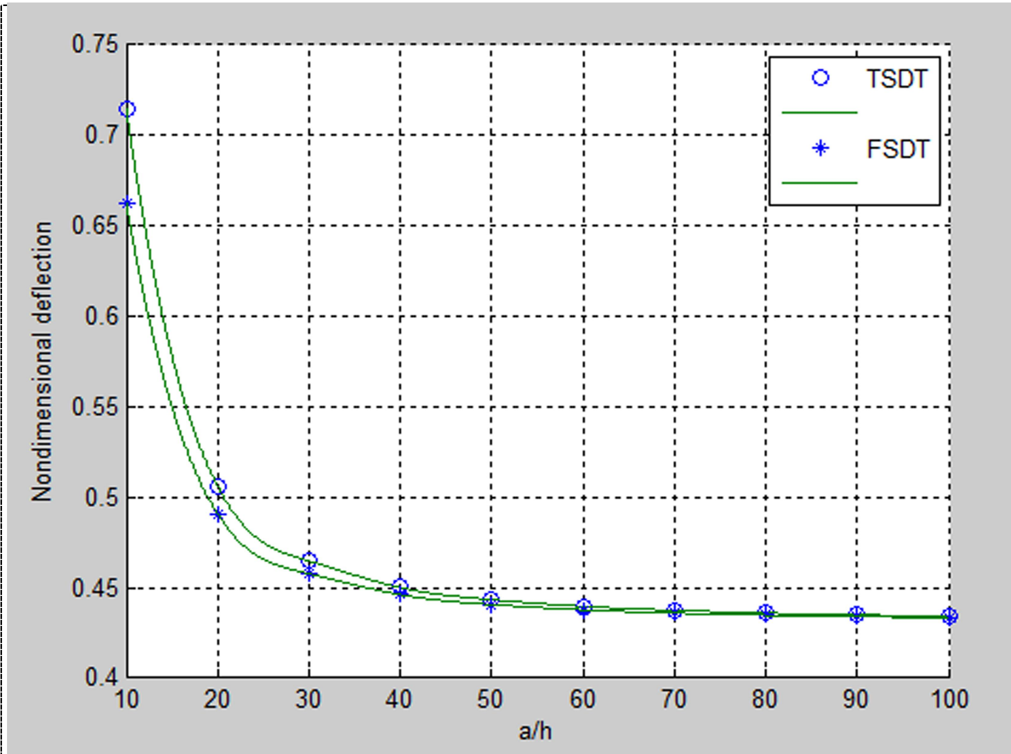

Figure 2. Plots of non-dimensionalized centre transverse deflection versus side-to-thickness ratio of a symmetric cross-ply (0/90/90/0) laminate under sinusoidal distributed load.

5. Conclusion

Detailed bending analysis of a laminated composite plate has been studied using FSDT and HSDT. Formulations based on both FSDT and HSDT explained in a detailed manner. Matlab codes for complete FSDT and HSDT analysis has done. The codes are providing satisfactory results when compared with references. The convergence is obtaining in composite plate analysis. The results of non-dimensionalized central transverse deflection various conditions are calculated and compared with published results available in literature. A very good agreement of the results obtained by present method with reference solutions, shows that the formulation and programming is robust, effective and highly accurate. As the *a/h* ratio is increased, the non-dimensional transverse deflection decreases. And it is observed that for *a/h* ratio greater than 40, the deflection becomes almost constant. The difference between FSDT and HSDT values are decreases as the thickness of plate decreases. Thus the current study elaborately discussed the FEM formulation that makes an easy programming even for a beginner in this field and presented some significant results for the structural responses of composite plates.

IOP Conf. Series: Materials Science and Engineering **225** (2017) 012018 doi:10.1088/1757-899X/225/1/012018

6. References

[1] Zienkiewicz O C 1971*The Finite Element Method* (Tata McGraw-Hill Publishing)

[2] Reddy J N 2004 *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis* (New York: CRC Press)

[3] Agarwal B D, Broutman L J and Chandrashekhara K 2006 *Analysis and Performance of Fiber Composites* (New Jersey; John Wiley & Sons)

[4] Jones R M 1998 *Mechanics of Composite Materials* (CRC press)

[5] Bhavikatti S S 2005 *Finite Element Analysis* (New Age International Publishers)

[6] Sreehari V M and Maiti D K 2015 Buckling and Post buckling Analysis of Laminated Composite Plates in Hygrothermal Environment Using an Inverse Hyperbolic Shear Deformation Theory *Comp. Struct*. ***129*** 250-255

[7] Chandrupatla T R and Belegundu A D 2002 *Introduction to FE in Engineering* (Prentice Hall)

[8] Ferreira A J M 2008 *MATLAB Codes for Finite Element Analysis: Solids and Structures* (Springer Science & Business Media)

[9] Cook R D, Malkus D S, Plesha M E and Robert J W 2007 *Concepts and Applications of Finite Element Analysis* (John Wiley and Sons)

[10] Kwon Y W and Bang H 1997 *The Finite Element Method using Matlab* (Boca Raton, FL; CRSC Press)