IOP Conf. Series: Materials Science and Engineering **225** (2017) 012035 doi:10.1088/1757-899X/225/1/012035

The Numerical Solution of the Equilibrium Problem for a Stretchable Elastic Beam[1].

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Abstract. The boundary value problem under consideration describes the equilibrium of an elastic beam that is stretched or contracted by specified forces. The left end of the beam is free of load, and the right end is rigidly lapped. To solve the problem numerically, an appropriate difference problem is constructed. Solving the difference problem, we obtain an approximate solution of the problem. We estimate the approximate solution of the stated problem.

1. Introduction

The following boundary-value problem is considered

$$\frac{d^4x}{dt^4}+a(t)x=f(t),\quad (0<t<l), \tag{1}$$

$$\frac{d^2x(0)}{dx^2}=0,\ \frac{d^3x(0)}{dt^3}=0,\ \frac{dx(1)}{dt}=0,\ x(1)=\alpha x(c),\ (0<c<1),\ \alpha<1, \tag{2}$$

where $a(t)\geq 0$ and $f(t)$ $(0\leq t\leq 1)$ are continuous functions.

It is known that the problem of the equilibrium of a thin rod is reduced to the solution of problem (1) - (2). For the numerical solution of problem (1) - (2), the finite difference method is applied. Problem (1) - (2), describes the equilibrium of an elastic beam that is stretchable or compressible as a function of the sign of the function along the axis by specified forces. The left end of the beam is free of load, and the right end is rigidly lapped.

2. Numerical solution of the problem

To solve the problem numerically, we construct the following uniform grid:
$\overline{w}_h=\{t_i=ih,\ i=0,1,\dots,n\}$, where $h=1/n$ is grid step and n is arbitrary natural number.

Denote by $\check{w}_h$ the set of the following grid points:

$$\overset{\vee}{w}_h = \left\{ t_{i+\frac{1}{2}} = \left(i + \frac{1}{2} \right) h,\ i = 0,1,\ldots,n-1 \right\}.$$

We introduce the following notations

$$x_{\overset{\vee}{t}} = \frac{1}{2}\left[x\left(t + \frac{h}{2} \right) + x\left(t - \frac{h}{2} \right) \right],$$
$$x_{\hat{t}} = \frac{1}{h}\left[x\left(t + \frac{h}{2} \right) - x\left(t - \frac{h}{2} \right) \right],$$
$$x_{\bar{t}} = \frac{1}{2h}\left[x(t+h) - x(t-h) \right].$$

It is not difficult to see that problem (1) - (2) can be replaced by the following problem:

$$L_h^{(4)} x(t) + a(t)x(t) = f(t) + O(h^2),\ t \in \overset{\vee}{w}_h \qquad (3)$$

$$\left. \begin{array}{l} l_h^{(0)} x(1) = O(h^2), \quad l_h^{(1)} x(1) = O(h^2), \\ l_h^{(2)} x(0) = O(h^2), \quad l_h^{(3)} x(0) = O(h^2), \end{array} \right\} \qquad (4)$$

where

$$L_h^{(4)} x(t) \equiv \frac{1}{h^4}[x(t+2h) - 4x(t+h) + 6x(t) - 4x(t-h) + x(t-2h)],$$
$$l_h^{(0)} x(t) \equiv \frac{1}{2}\left[x\left(t + \frac{h}{2} \right) + x\left(t - \frac{h}{2} \right) \right],$$
$$l_h^{(1)} x(t) \equiv \frac{1}{4h}\left[x\left(t + \frac{3h}{2} \right) + x\left(t + \frac{h}{2} \right) - x\left(t - \frac{h}{2} \right) - x\left(t - \frac{3h}{2} \right) \right],$$

$$l_h^{(2)} x(t) \equiv \frac{1}{8h^2}\left[x\left(t + \frac{5h}{2} \right) + x\left(t + \frac{3h}{2} \right) - 2x\left(t + \frac{h}{2} \right) - \right.$$
$$\left. - 2x\left(t - \frac{h}{2} \right) + x\left(t - \frac{3h}{2} \right) + x\left(t - \frac{5h}{2} \right) \right],$$
$$l_h^{(3)} x(t) \equiv \frac{1}{16h^3}\left[x\left(t + \frac{7h}{2} \right) + 3x\left(t - \frac{3h}{2} \right) - x\left(t - \frac{5h}{2} \right) - x\left(t - \frac{7h}{2} \right) \right].$$

However, the problem (3) - (4) can be replaced by the following system

$$x_{i+\frac{5}{2}} - 4x_{i+\frac{3}{2}} + \left(6 + h^4 a_{i+\frac{1}{2}} \right) x_{i+\frac{1}{2}} - 4x_{i-\frac{1}{2}} +$$
$$+ x_{i-\frac{3}{2}} = h^4 f_{i+\frac{1}{2}} + O(h^4),\ (i = 0,1,\ldots,n-1) \qquad (5)$$

$$\left.\begin{aligned}
&x_{n+\frac{1}{2}} + x_{n-\frac{1}{2}} = O(h^2),\ x_{n+\frac{3}{2}} - x_{n-\frac{3}{2}} = O(h^3),\\
&x_{\frac{3}{2}} - x_{\frac{1}{2}} - x_{-\frac{1}{2}} + x_{-\frac{3}{2}} = O(h^4),\\
&x_{\frac{3}{2}} - 3x_{\frac{1}{2}} + 3x_{-\frac{1}{2}} - x_{-\frac{3}{2}} = O(h^5).
\end{aligned}\right\} \tag{6}$$

From (6) we obtain that

$$\begin{aligned}
&x_{n+\frac{1}{2}} = -x_{n-\frac{1}{2}} + O(h^2),\\
&x_{n+\frac{3}{2}} = x_{n-\frac{3}{2}} + O(h^3),\\
&x_{-\frac{1}{2}} = -x_{\frac{3}{2}} + 2x_{\frac{1}{2}} + O(h^4),\\
&x_{-\frac{3}{2}} = -2x_{\frac{3}{2}} + 3x_{\frac{1}{2}} + O(h^4).
\end{aligned}$$

Taking these formulas into account (5), we get

$$\left(5 + h^4 a_{\frac{1}{2}}\right) x_{\frac{1}{2}} - 6x_{\frac{3}{2}} + x_{\frac{5}{2}} = h^4 f_{\frac{1}{2}} + O(h^4),$$

$$-2x_{\frac{1}{2}} + \left(5 + h^4 a_{\frac{3}{2}}\right) x_{\frac{3}{2}} - 4x_{\frac{5}{2}} + x_{\frac{7}{2}} = h^4 f_{\frac{3}{2}} + O(h^4),$$

$$x_{\frac{1}{2}} - 4x_{\frac{3}{2}} + \left(6 + h^4 a_{\frac{5}{2}}\right) x_{\frac{5}{2}} - 4x_{\frac{7}{2}} + x_{\frac{9}{2}} = h^4 f_{\frac{5}{2}} + O(h^4),$$

....

$$\begin{aligned}
&x_{n-\frac{9}{2}} - 4x_{n-\frac{7}{2}} + \left(6 + h^4 a_{n-\frac{5}{2}}\right) x_{n-\frac{5}{2}} - 4x_{n-\frac{3}{2}} + x_{n-\frac{1}{2}} =\\
&= h^4 f_{n-\frac{5}{2}} + O(h^4),
\end{aligned}$$

$$\begin{aligned}
&x_{n-\frac{7}{2}} - 4x_{n-\frac{5}{2}} + \left(6 + h^4 a_{n-\frac{3}{2}}\right) x_{n-\frac{3}{2}} - 4x_{n-\frac{1}{2}} + x_{n+\frac{1}{2}} =\\
&= h^4 f_{n-\frac{3}{2}} + O(h^4),
\end{aligned}$$

$$\begin{aligned}
&x_{n-\frac{5}{2}} - 4x_{n-\frac{3}{2}} + \left(6 + h^4 a_{n-\frac{1}{2}}\right) x_{n-\frac{1}{2}} - 4x_{n+\frac{1}{2}} + x_{n+\frac{3}{2}} =\\
&= h^4 f_{n-\frac{1}{2}} + O(h^4),\\
&x_{n+\frac{1}{2}} = -x_{n-\frac{1}{2}} + O(h^2),\ x_{n+\frac{1}{2}} = -x_{n-\frac{3}{2}} + O(h^3).
\end{aligned}$$

Discarding the right-hand sides, we obtain the following difference scheme for problem (1)-(2):

$$\left.\begin{array}{l}
\left(5+h^4 a_{\frac{1}{2}}\right)x_{\frac{1}{2}}-6x_{\frac{3}{2}}+x_{\frac{5}{2}}=h^4 f_{\frac{1}{2}}, \\
-2x_{\frac{1}{2}}+\left(5+h^4 a_{\frac{3}{2}}\right)x_{\frac{3}{2}}-4x_{\frac{5}{2}}+x_{\frac{7}{2}}=h^4 f_{\frac{3}{2}}, \\
x_{\frac{1}{2}}-4x_{\frac{3}{2}}+\left(6+h^4 a_{\frac{5}{2}}\right)x_{\frac{5}{2}}-x_{\frac{7}{2}}+x_{\frac{9}{2}}=h^4 f_{\frac{5}{2}}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x_{n-\frac{5}{2}}-4x_{n-\frac{3}{2}}+\left(6+h^4 a_{n-\frac{1}{2}}\right)x_{n-\frac{1}{2}}-4x_{n+\frac{1}{2}}+ \\
+x_{n+\frac{3}{2}}=h^4 f_{n-\frac{1}{2}}, \\
x_{n+\frac{1}{2}}=-x_{n-\frac{1}{2}},\ x_{n+\frac{3}{2}}=x_{n-\frac{3}{2}}.
\end{array}\right\} \tag{7}$$

The solution of the linear system (7) we will take as the approximate solution of the problem (1) - (2) at the points $\overset{\vee}{w}_h$. Error estimation of the approximate solution of problem (1) - (2) of $O(h^2)$ order.

3. Increasing the accuracy of solving difference schemes.

For any sufficiently smooth function $\varphi(t)$ at $t \in \overset{\vee}{w}_h$ takes place

$$L^{(4)}\varphi + a\varphi = \varphi^{(4)}(t) + a(t)\varphi(t) + 8\sum_{i=1}^{s}\frac{\varphi^{(4+2i)}(t)}{(4+2i)!}(2^{2i+2}-1)h^{2i} + O(h^{2s+2})$$

while $t \in \overline{\omega}_h$

$$l^{(0)}\varphi(t) = \varphi(t) + \sum_{i=1}^{s}\left(\frac{h}{2}\right)^{2i}\frac{\varphi^{(2i)}(t)}{(2i)!} + O(h^{2s+2}),$$

$$l^{(1)}\varphi(t) = \varphi'(t) + \sum_{i=1}^{s}\frac{h^{2i}}{2^{2(i+1)}}\cdot(1+3^{2i+1}) + O(h^{2s+2}),$$

$$l^{(2)}\varphi(t) = \varphi''(t) + \sum_{i=1}^{s}\frac{h^{2i}}{2^{2(i+2)}}\cdot\frac{\varphi^{(2i+2)}(t)}{(2i+2)!}\cdot(5^{2(i+1)}+3^{2(i+1)}-2) + O(h^{2s+2}),$$

$$l^{(3)}\varphi(t) = \varphi'''(t) + \sum_{i=1}^{s}\frac{h^{2i}}{2^{2(i+3)}}\cdot\frac{\varphi^{(2i+3)}(t)}{(2i+3)!}\cdot(7^{2i+3}+5^{2i+3}-3^{2i+4}-3) + O(h^{2s+2}).$$

Taking these formulas into account, we prove that the solution of (7) can be represented as follows:

$$x^k = \sum_{j=0}^{j-1} h^{2j} Y_j(t_k) + \eta_k^h \quad (k = 0,1,...,n), \tag{8}$$

where

$$\left\|\eta^h\right\|_{L^2(\overset{\vee}{w}_h)} \le ch^{2l} \quad (c = const,\ l = 2,3,...). \tag{9}$$

Using the method of increasing accuracy, the accuracy of solving problem (1) - (2) increases to the order of $O(h^{2l})$.

References

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